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A STUDY OF THE DISTRIBUTION OF MOLECULES UNDER FREE MOLECULAR FLOW CONDITIONS AFTER COLLISIONS WITH SIMPLE GEOMETRIES

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ABSTRACT

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The study makes a detailed examination of molecular distribution under free molecular flow conditions within surfaces of simple geometries. By considering infinite length in one dimension, the problem could be reduced to a two-dimensional analysis. Exact, closed form solutions were obtained only for circles; however, numerical solutions were obtained for ellipses and parabolas. Monte Carlo computer technique results were obtained and compared to the theoretical solutions. Derivations of (a) the distribution of molecules over the surface as a function of collision number, (b) the exit distribution across the opening of each surface configuration, and (c) the distribution across the center line of each configuration are presented.

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Any errors are, of course, the responsibility of the authors and they would be pleased to receive any comments concerning the paper.

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DEFINITION OF SYMBOLS

Symbols	Definition	
a	$x_0 - x_1$	[68]
A	variable parameter in equation for ellipse	[43]
b	distance from P_α to point (0, z) on Y-axis	[68]
c	$x_0 - x_2$	[68]
d	distance from P_α to point $(x_1, 0)$	[68]
$F^{(n)}(\alpha, \beta)$	$1/2 \int N_\alpha^{(n-1)} (\cos\frac{\alpha}{2} \cos\frac{\beta}{2} + \sin\frac{\alpha}{2} \sin\frac{\beta}{2})\, d\alpha$	[12]
n	collision number	[11]
N_α	number of molecules per radian reflected from the surface at P_α	[3]
$N_\alpha^{(n)}$	number of molecules per radian reflected from P_α on n^{th} collision	[3]
$N_{\alpha,\beta}$	number of molecules per radian from P_α that hit section of surface between $\alpha = \beta$ and $\alpha = \pi/2$	[4]
$N_{\alpha,\varphi}$	number of molecules per radian reflected from P_α at an angle φ with the normal	[3]
N_β	total number of molecules that hit section of surface between $\alpha = \beta$ and $\alpha = \pi/2$	[10]
$N_\beta^{(n)}$	total number of molecules that hit section of surface between $\alpha = \beta$ and $\alpha = \pi/2$ on the n^{th} collision	[5]
N_x	total number of molecules that pass through a section of the center line between $(x_1, 0)$ and $(x_2, 0)$ from all points on the surface	[73]

NOTE: Numbers in brackets refer to page on which symbol is first used.

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition	
N_z	number of molecules that exit an array between (0, 0) and (0, z)	[52]
$N_{z,\alpha}$	number of molecules per radian from P_α that exit the array between (0, 0) and (0, z)	[51]
N_o	total number of incident molecules	[3]
P	variable parameter in equation for parabola	[46]
P_α	point with coordinates (x, y) at angle α	[2]
r	distance from P_α to point (x_2, 0)	[68]
s	distance from point (0, 0) to P_α	[66]
x_o	intersection of normal with x-axis	[66]
x_1, x_2	points on the center line which define the section of interest for distribution of molecules across the center line	[66]
x, y	coordinates of arbitrary point P_α on surface	[2]
y_o	intersection of normal with y-axis	[54]
Z	value of y which defines the area of interest for exit distribution across the y-axis	[51]
α	angle from negative y-axis to point (x, y) on reflecting surface	[2]
α_n	ratio of number of molecules that exit an array after n collisions to the number that make n collisions	[77]
β	angle from negative y-axis to section of interest on surface	[2]
δ, ϵ	angles <u>from</u> normal to P_α <u>to</u> limits of molecule beam to section of interest	[2]
φ	general angle <u>from</u> the normal to the surface at P_α <u>to</u> the path of a reflected molecule	[3]
θ	angle from negative x-axis to tangent at P_α	[66]

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SUMMARY

The study makes a detailed examination of molecular distribution under free molecular flow conditions within surfaces of simple geometries. By considering infinite length in one dimension, the problem could be reduced to a two-dimensional analysis. Exact, closed form solutions were obtained only for circles; however, numerical solutions were obtained for ellipses and parabolas. Monte Carlo computer technique results were obtained and compared to the theoretical solutions. Derivations of (a) the distribution of molecules over the surface as a function of collision number, (b) the exit distribution across the opening of each surface configuration, and (c) the distribution across the center line of each configuration are presented.

SECTION I. INTRODUCTION

Studies of free molecular flow problems have usually been concerned with the gross effects resulting from the flow such as the conductance of a tube or the forces on a body moving at a velocity large compared to the thermal velocity of the gas molecules. The analytical solutions are usually obtained in exact forms for limited geometries. The use of high speed digital computer methods [1, 2, 3, 4] has allowed the determination of many interesting parameters such as distribution of molecules, average number of collisions within surfaces, etc., parameters which would have been difficult if not impossible to determine from exact analytical studies. It was felt by the authors that a detailed examination of molecules under free molecular flow conditions would allow much greater insight into actual physical processes and would assist in practical problems such as cryogenic pump array design, molecule-surface interaction studies, high altitude atmospheric measuring programs, etc.

Typical of most studies of free molecular flow, two assumptions have been made: (1) The molecules interact only with the surfaces and (2) molecules colliding with the wall are re-emitted diffusely. (The terms "molecules" and "particles" are used interchangeably throughout.) In all cases, the surfaces examined are considered infinitely long in the "z" direction such that the analysis can be reduced to two dimensions. Theoretical studies presented in this paper consider only elementary surfaces which, in two dimensions, are circles, ellipses, and parabolas; however, completely digital computer studies have considered hyperbolas and triangles [3]. Only for the case of the circle could solutions be made exactly in closed form. The other cases resulted in integrals, some of which were solved by computer methods. Monte Carlo methods used for this study are described in Reference 3.

This paper presents the derivation of (a) the distribution of molecules over the surface as a function of collision number, (b) the exit distribution across the opening of each surface configuration, and (c) the distribution across the center line of each configuration. The results of numerically integrating these expressions are given and compared with the results obtained by the Monte Carlo method.

SECTION II. DISTRIBUTION OF MOLECULES OVER A SEMICIRCULAR SURFACE AS A FUNCTION OF COLLISION NUMBER

A. General Expressions

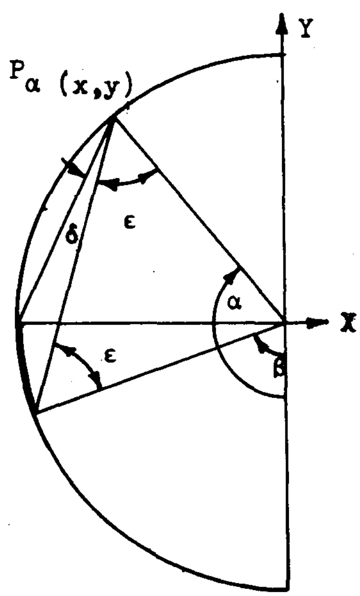

FIGURE 1

Consider a flux of molecules entering the semicircular geometry shown in Figure 1. After being reflected at the surface, some of the molecules will exit the array, while others will collide again with the surface a number of times before exiting.

The problem to be considered here is to find expressions for the distribution of these molecules over the surface, as a function of collision number. When the initial distribution is known, the problem is simply a matter of trigonometry and calculus, as follows:

Let the initial distribution be given by

$N_\alpha^{(1)}$ = number of molecules per radian that collide with the surface at point P_α on the first collision. (See Figure 1.)

Then the total number of incident molecules, N_o, is given by

$$N_o = \int_o^\pi N_\alpha^{(1)}\, d\alpha.$$

According to Lambert's Cosine Law of Diffuse Reflection, the number of molecules per radian that are reflected from P_α at an angle φ with the normal is proportional to the cosine of this angle, i.e.,

$$N_{\alpha,\varphi} = k \cos \varphi.$$

Then clearly,

$$N_\alpha = \int_{-\frac{\pi}{2}}^{\pi/2} k \cos \varphi \, d\varphi,$$

where N_α is the surface distribution function for any arbitrary collision. This yields immediately

$$k = \frac{1}{2} N_\alpha,$$

so that

$$N_{\alpha,\varphi} = \frac{1}{2} N_\alpha \cos \varphi.$$

Now the number of molecules that collide again with a section of the surface between β and $\pi/2$ (shaded band in Figure 1) will be the sum of three contributions: (1) for $\alpha > \pi/2$, (2) for $\pi/2 \geq \alpha \geq \beta$, and (3) for $\beta > \alpha$.

Contribution (1): $\alpha > \pi/2$

The number of molecules per radian reflected from P_α that hit between β and $\pi/2$ is

$$N^1_{\alpha,\beta} = \int_\epsilon^{\epsilon+\delta} \frac{1}{2} N_\alpha \cos \varphi \, d\varphi = \frac{1}{2} N_\alpha \, [\sin \varphi]_\epsilon^{\epsilon+\delta}$$

$$N^1_{\alpha,\beta} = \frac{1}{2} N_\alpha \left[\sin (\epsilon + \delta) - \sin \epsilon\right]$$

From Figure 1:

$$2\epsilon = \pi - (\alpha - \beta), \qquad \text{or} \qquad \epsilon = \frac{\pi}{2} - \left(\frac{\alpha - \beta}{2}\right),$$

also

$$2(\epsilon + \delta) = \pi - \left(\alpha - \frac{\pi}{2}\right), \qquad \text{or} \qquad \epsilon + \delta = \frac{\pi}{2} - \left(\frac{\alpha}{2} - \frac{\pi}{4}\right).$$

Hence,

$$\sin (\epsilon + \delta) = \sin \left[\frac{\pi}{2} - \left(\frac{\alpha}{2} - \frac{\pi}{4}\right)\right] = \cos \left(\frac{\alpha}{2} - \frac{\pi}{4}\right),$$

and

$$\sin \epsilon = \sin \left[\frac{\pi}{2} - \left(\frac{\alpha - \beta}{2}\right)\right] = \cos\left(\frac{\alpha - \beta}{2}\right),$$

so that

$$N^1_{\alpha,\beta} = \frac{1}{2} N_\alpha \left[\cos\left(\frac{\alpha}{2} - \frac{\pi}{4}\right) - \cos\left(\frac{\alpha - \beta}{2}\right)\right].$$

Then the total number of molecules that hit between β and $\pi/2$, from all P_α, with $\alpha \geq \pi/2$, is

$$N^1_\beta = \int_{\frac{\pi}{2}}^{\pi} N^1_{\alpha,\beta}\, d\alpha = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} N_\alpha \left[\cos\left(\frac{\alpha}{2} - \frac{\pi}{4}\right) - \cos\left(\frac{\alpha - \beta}{2}\right)\right] d\alpha \; . \; . \; . \qquad (1)$$

Contribution (2): $\frac{\pi}{2} \geq \alpha \geq \beta$

In this case there will be two contributions: (2-1), angles greater than α, and (2-2), angles less than α (Fig. 2). For the first contribution (2-1):

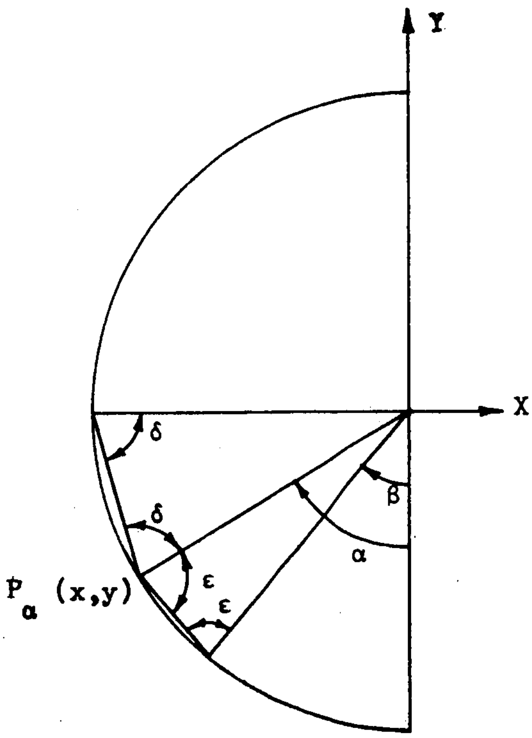

Figure 2

$$N_{\alpha,\beta}^{2-1} = \int_{-\frac{\pi}{2}}^{-\delta} \frac{1}{2} N_\alpha \cos \varphi \, d\varphi = \frac{1}{2} N_\alpha \left[\sin(-\delta) - \sin\left(-\frac{\pi}{2}\right)\right],$$

$$N_{\alpha,\beta}^{2-1} = \frac{1}{2} N_\alpha (1 - \sin \delta),$$

but, from Figure 2:

$$2\delta = \pi - \left(\frac{\pi}{2} - \alpha\right) = \frac{\pi}{2} + \alpha,$$

or

$$\delta = \frac{\pi}{4} + \frac{\alpha}{2}$$

so that

$$N_{\alpha,\beta}^{2-1} = \frac{1}{2} N_\alpha \left[1 - \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right].$$

For the second contribution:

$$N_{\alpha,\beta}^{2-2} = \int_{\epsilon}^{\pi/2} \frac{1}{2} N_\alpha \cos \varphi \, d\varphi = \frac{1}{2} N_\alpha \left(\sin \frac{\pi}{2} - \sin \epsilon\right),$$

$$N_{\alpha,\beta}^{2-2} = \frac{1}{2} N_\alpha (1 - \sin \epsilon),$$

but

$$2\epsilon = \pi - (\alpha - \beta)$$

or

$$\epsilon = \frac{\pi}{2} - \left(\frac{\alpha - \beta}{2}\right),$$

so that

$$N^{2-2}_{\alpha,\beta} = \frac{1}{2} N_\alpha \left[1 - \sin\left(\frac{\pi}{2} - \frac{\alpha - \beta}{2}\right)\right] = \frac{1}{2} N_\alpha \left[1 - \cos\left(\frac{\alpha - \beta}{2}\right)\right].$$

Therefore,

$$N^{2}_{\alpha,\beta} = N^{2-1}_{\alpha,\beta} + N^{2-2}_{\alpha,\beta} = \frac{1}{2} N_\alpha \left[2 - \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \cos\left(\frac{\alpha - \beta}{2}\right)\right].$$

Then the <u>total</u> number of molecules that hit between β and $\pi/2$, from all P_α, such that $\pi/2 \geq \alpha \geq \beta$ is

$$N^{2}_{\beta} = \int_{\beta}^{\pi/2} \frac{1}{2} N_\alpha \left[2 - \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \cos\left(\frac{\alpha - \beta}{2}\right)\right] d\alpha \; . \; . \; . \qquad (2)$$

<u>Contribution (3): $\alpha \leq \beta$</u>

From Figure 3:

$$N^{3}_{\alpha,\beta} = \int_{\epsilon}^{\epsilon+\delta} \frac{1}{2} N_\alpha \cos\varphi \, d\varphi,$$

$$N^{3}_{\alpha,\beta} = \frac{1}{2} N_\alpha (\sin(\epsilon+\delta) - \sin\epsilon).$$

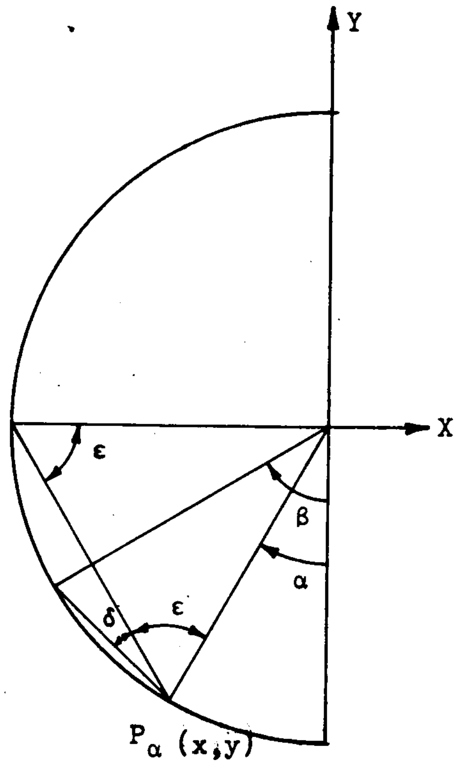

FIGURE 3

Now

$$2(\epsilon+\delta) = \pi - (\beta-\alpha)$$

$$\epsilon + \delta = \frac{\pi}{2} - \left(\frac{\beta-\alpha}{2}\right)$$

$$\sin(\epsilon+\delta) = \cos\left(\frac{\beta-\alpha}{2}\right).$$

Also

$$2\epsilon = \pi - \left(\frac{\pi}{2} - \alpha\right)$$

$$\epsilon = \frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = \frac{\pi}{4} + \frac{\alpha}{2}$$

$$\sin\epsilon = \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

so that

$$N^{(3)}_{\alpha,\beta} = \frac{1}{2} N_\alpha \left[\cos\left(\frac{\beta - \alpha}{2}\right) - \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right]$$

and the total number of molecules that hit between β and $\pi/2$, from <u>all</u> $P\alpha$, such that $\beta \geq \alpha$, is

$$N^3_\beta = \int_0^\beta \frac{1}{2} N_\alpha \left[\cos\left(\frac{\alpha - \beta}{2}\right) - \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right] d\alpha. \tag{3}$$

Combining equations (1), (2) and (3), the <u>total</u> contribution to the number of molecules hitting between β and $\pi/2$ from all $P\alpha$, is:

$$N_\beta = \int_{\pi/2}^{\pi} \frac{1}{2} N_\alpha \left[\underset{(1)}{\cos\left(\frac{\alpha}{2} - \frac{\pi}{4}\right)} - \underset{(2)}{\cos\left(\frac{\alpha - \beta}{2}\right)}\right] d\alpha\, +$$

$$+ \int_{\beta}^{\pi/2} \frac{1}{2} N_\alpha \left[2 - \underset{(3)}{\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)} - \underset{(4)}{\cos\left(\frac{\alpha - \beta}{2}\right)}\right] d\alpha\, +$$

$$+ \int_0^\beta \frac{1}{2} N_\alpha \left[\underset{(5)}{\cos\left(\frac{\alpha - \beta}{2}\right)} - \underset{(6)}{\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)}\right] d\alpha.$$

Now terms (2) and (4), and terms (3) and (6) can be combined directly. Also, since

$$\cos\left(\frac{\alpha}{2} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right),$$

the expression reduces to

$$N_\beta = \int_{\pi/2}^{\pi} \frac{1}{2} N_\alpha \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) d\alpha - \int_{\beta}^{\pi} \frac{1}{2} N_\alpha \cos\left(\frac{\alpha - \beta}{2}\right) d\alpha +$$

$$+ \int_{\beta}^{\pi/2} N_\alpha d_\alpha - \int_{o}^{\pi/2} \frac{1}{2} N_\alpha \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) d\alpha + \int_{o}^{\beta} \frac{1}{2} N_\alpha \cos\left(\frac{\alpha - \beta}{2}\right) d\alpha.$$

Now, from the symmetry of the problem (Figure 1), it is clear that

$$N_{\pi-\alpha} = N_\alpha \cdot$$

Also

$$\sin\left(\frac{\pi}{4} + \frac{\pi - \alpha}{2}\right) = \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

and

$$d(\pi - \alpha) = - d\alpha,$$

so that by substituting $(\pi - \alpha)$ for α in the first integral:

$$\int_{\pi/2}^{\pi} \frac{1}{2} N_\alpha \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) d\alpha = \int_{\pi/2}^{o} \frac{1}{2} N_{\pi-\alpha} \sin\left(\frac{\pi}{4} + \frac{\pi - \alpha}{2}\right) d(\pi - \alpha)$$

$$= \int_{o}^{\pi/2} \frac{1}{2} N_\alpha \sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) d\alpha,$$

so that it cancels with the fourth integral. With this simplification, the expression becomes

$$N_\beta = -\int_\beta^\pi \frac{1}{2} N_\alpha \cos\left(\frac{\alpha - \beta}{2}\right) d\alpha + \int_\beta^{\pi/2} N_\alpha \, d\alpha + \int_0^\beta \frac{1}{2} N_\alpha \cos\left(\frac{\alpha - \beta}{2}\right) d\alpha. \qquad (4)$$

This is the total number of molecules that hit between β and $\pi/2$ from all points on the surface. Since this will vary with the collision number, n, let $N_\beta^{(n)}$ = total number of molecules that hit the surface between β and $\pi/2$ on the n^{th} collision.

Now the number of molecules per radian that hit a <u>point</u>, P_α, on the n^{th} collision is found by calculating the limit:

$$\lim_{\Delta\beta \to 0} \left[\frac{N_\beta^{(n)} - N_{\beta+\Delta\beta}^{(n)}}{\Delta\beta}\right].$$

But this is clearly just the derivative $\frac{-d}{d\beta} N_\beta^{(n)}$; therefore,

$$N_\alpha^{(n)} = \left.\frac{-d}{d\beta} N_\beta^{(n)}\right]_{\beta=\alpha},$$

so that

$$\int_\beta^{\pi/2} N_\alpha^{(n-1)} \, d\alpha = \left[-N_\beta^{(n-1)}\right]_\beta^{\pi/2}$$

From Figure 1 it is clear that $N_\beta^{(n-1)} = 0$ for $\beta = \pi/2$ and any n. Therefore,

$$\int_\beta^{\pi/2} N_\alpha^{(n-1)} \, d\alpha = N_\beta^{(n-1)}. \qquad (5)$$

Also,

$$\cos\left(\frac{\alpha - \beta}{2}\right) = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2} \,. \tag{6}$$

Now, let

$$F^{(n)}(\alpha, \beta) \equiv \frac{1}{2}\int N_\alpha^{(n-1)}\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) d\alpha . \tag{7}$$

Then, using equations (5), (6), and (7) in equation (4), it follows that

$$N_\beta^{(n)} = - F^{(n)}(\pi, \beta) + F^{(n)}(\beta, \beta) + N_\beta^{(n-1)} + F^{(n)}(\beta, \beta) - F^{(n)}(0, \beta),$$

or

$$N_\beta^{(n)} = 2F^{(n)}(\beta, \beta) - F^{(n)}(0, \beta) - F^{(n)}(\pi, \beta) + N_\beta^{(n-1)} \tag{8}$$

and

$$N_\alpha^{(n)} = - \frac{d}{d\beta} N_\beta^{(n)}\Big]_{\beta = \alpha} \quad , \tag{9}$$

which is the final expression for the distribution. Given any initial distribution, $N_\alpha^{(1)}$, the distribution at the next collision can be found by first calculating $F^{(2)}(\alpha, \beta)$ from equation (7), using this expression (with the indicated substitutions) in equation (8), and then using equation (9).

For specific cases, the elementary integrals given in Appendix A will be useful.

B. Case I: Incident Distribution Uniform over the Surface

1. Distribution at Second Collision

For an incident distribution of molecules uniform over the surface,

$$N_{\alpha}^{(1)} = \frac{N_o}{\pi} = \text{constant}, \tag{10}$$

where N_o is the total number of incident molecules. Then, from equation (5),

$$N_{\beta}^{(1)} = \int_{\beta}^{\pi/2} N_{\alpha}^{(1)} \, d\alpha = \int_{\beta}^{\pi/2} \frac{N_o}{\pi} \, d\alpha,$$

$$N_{\beta}^{(1)} = \frac{N_o}{\pi}\left(\frac{\pi}{2} - \beta\right). \tag{11}$$

From equations (7) and (10),

$$F^{(2)}(\alpha, \beta) \doteq \frac{1}{2} \int \frac{N_o}{\pi}\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) d\alpha,$$

$$F^{(2)}(\alpha, \beta) = \frac{N_o}{2\pi}\left(2\sin\frac{\alpha}{2}\cos\frac{\beta}{2} - 2\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right).$$

Then,

$$F^{(2)}(\beta, \beta) = \frac{N_o}{2\pi}\left(2\sin\frac{\beta}{2}\cos\frac{\beta}{2} - 2\cos\frac{\beta}{2}\sin\frac{\beta}{2}\right) = 0,$$

$$F^{(2)}(0, \beta) = \frac{N_o}{2\pi}\left(-2\sin\frac{\beta}{2}\right) = -\frac{N_o}{\pi}\sin\frac{\beta}{2},$$

$$F^{(2)}(\pi, \beta) = \frac{N_o}{2\pi}\left(2\cos\frac{\beta}{2}\right) = \frac{N_o}{\pi}\cos\frac{\beta}{2},$$

and, using equations (8) and (11):

$$N_{\beta}^{(2)} = 2F^{(2)}(\beta, \beta) - F^{(2)}(0, \beta) - F^{(2)}(\pi, \beta) + N_{\beta}^{(1)}$$

$$= \frac{N_o}{\pi} \sin \frac{\beta}{2} - \frac{N_o}{\pi} \cos \frac{\beta}{2} + \frac{N_o}{\pi}\left(\frac{\pi}{2} - \beta\right).$$

$$N_{\beta}^{(2)} = \frac{N_o}{\pi}\left(\frac{\pi}{2} - \beta + \sin \frac{\beta}{2} - \cos \frac{\beta}{2}\right). \tag{12}$$

From equation (9):

$$N_{\alpha}^{(2)} = - \frac{d}{d\beta} N_{\beta}^{(2)}\Bigg]_{\beta=\alpha} = - \frac{N_o}{\pi}\left(-1 + \frac{1}{2} \cos \frac{\beta}{2} + \frac{1}{2} \sin \frac{\beta}{2}\right)\Bigg]_{\beta=\alpha},$$

$$N_{\alpha}^{(2)} = \frac{N_o}{2\pi}\left(2 - \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}\right) \tag{13}$$

2. Third Collision

From equations (7) and (13):

$$F^{(3)}(\alpha, \beta) = \frac{N_o}{4\pi} \int \left(2 - \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}\right)\left(\cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2}\right) d\alpha,$$

$$F^{(3)}(\alpha, \beta) = \frac{N_o}{4\pi}\left[4 \sin \frac{\alpha}{2} \cos \frac{\beta}{2} - 4 \cos \frac{\alpha}{2} \sin \frac{\beta}{2} - \left(\frac{\alpha}{2} + \frac{1}{2} \sin \alpha\right) \cos \frac{\beta}{2}\right.$$

$$\left. - \sin^2 \frac{\alpha}{2} \sin \frac{\beta}{2} - \sin^2 \frac{\alpha}{2} \cos \frac{\beta}{2} - \left(\frac{\alpha}{2} - \frac{1}{2} \sin \alpha\right) \sin \frac{\beta}{2}\right].$$

Then,

$$F^{(3)}(\beta, \beta) = \frac{N_o}{4\pi}\left[-\left(\frac{\beta}{2} + \frac{1}{2}\sin\beta\right)\cos\frac{\beta}{2} - \sin^2\frac{\beta}{2}\sin\frac{\beta}{2} - \sin^2\frac{\beta}{2}\cos\frac{\beta}{2}\right.$$

$$\left. - \left(\frac{\beta}{2} - \frac{1}{2}\sin\beta\right)\sin\frac{\beta}{2}\right]$$

$$= \frac{N_o}{4\pi}\left[-\left(\frac{\beta}{2} + \frac{1}{2}\sin\beta\right)\cos\frac{\beta}{2} - \sin\frac{\beta}{2} + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\right.$$

$$\left. - \sin^2\frac{\beta}{2}\cos\frac{\beta}{2} - \left(\frac{\beta}{2} - \frac{1}{2}\sin\beta\right)\sin\frac{\beta}{2}\right]$$

$$= \frac{N_o}{4\pi}\left[-\left(\frac{\beta}{2} + \frac{1}{2}\sin\beta\right)\cos\frac{\beta}{2} - \sin\frac{\beta}{2} + \frac{1}{2}\cos\frac{\beta}{2}\sin\beta\right.$$

$$\left. - \frac{1}{2}\sin\frac{\beta}{2}\sin\beta - \left(\frac{\beta}{2} - \frac{1}{2}\sin\beta\right)\sin\frac{\beta}{2}\right].$$

$$F^{(3)}(\beta, \beta) = \frac{N_o}{4\pi}\left[-\frac{\beta}{2}\left(\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\right) - \sin\frac{\beta}{2}\right].$$

$$F^{(3)}(0, \beta) = \frac{N_o}{4\pi}\left(-4\sin\frac{\beta}{2}\right).$$

$$F^{(3)}(\pi, \beta) = \frac{N_o}{4\pi}\left(4\cos\frac{\beta}{2} - \frac{\pi}{2}\cos\frac{\beta}{2} - \sin\frac{\beta}{2} - \cos\frac{\beta}{2} - \frac{\pi}{2}\sin\frac{\beta}{2}\right).$$

Then, using equations (8) and (12):

$$N_\beta^{(3)} = 2F^{(3)}(\beta, \beta) - F^{(3)}(0, \beta) - F^{(3)}(\pi, \beta) + N_\beta^{(2)}$$

$$= \frac{N_o}{4\pi}\left[-\beta\left(\cos\frac{\beta}{2}+\sin\frac{\beta}{2}\right)-2\sin\frac{\beta}{2}+4\sin\frac{\beta}{2}-4\cos\frac{\beta}{2}\right.$$

$$\left.+\frac{\pi}{2}\left(\cos\frac{\beta}{2}+\sin\frac{\beta}{2}\right)+\cos\frac{\beta}{2}+\sin\frac{\beta}{2}+4\sin\frac{\beta}{2}-4\cos\frac{\beta}{2}+2\pi-4\beta\right].$$

$$N_\beta^{(3)} = \frac{N_o}{4\pi}\left[2\pi-4\beta+\left(\frac{\pi}{2}-\beta+7\right)\sin\frac{\beta}{2}+\left(\frac{\pi}{2}-\beta-7\right)\cos\frac{\beta}{2}\right] \tag{14}$$

From equation (9):

$$N_\alpha^{(3)} = -\frac{d}{d\beta}N_\beta^{(3)}\Big]_{\beta=\alpha}$$

$$= -\frac{N_o}{4\pi}\left[-4+\frac{1}{2}\left(\frac{\pi}{2}-\beta+7\right)\cos\frac{\beta}{2}-\frac{1}{2}\left(\frac{\pi}{2}-\beta-7\right)\sin\frac{\beta}{2}-\sin\frac{\beta}{2}-\cos\frac{\beta}{2}\right]_{\beta=\alpha}$$

$$N_\alpha^{(3)} = \frac{N_o}{8\pi}\left[8-\left(\frac{\pi}{2}-\alpha+5\right)\cos\frac{\alpha}{2}+\left(\frac{\pi}{2}-\alpha-5\right)\sin\frac{\alpha}{2}\right]. \tag{15}$$

3. Fourth Collision

From equations (7) and (15):

$$F^{(4)}(\alpha,\beta) = \frac{N_o}{16\pi}\int\left[8-\left(\frac{\pi}{2}-\alpha+5\right)\cos\frac{\alpha}{2}+\left(\frac{\pi}{2}-\alpha-5\right)\sin\frac{\alpha}{2}\right]$$

$$\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)d\alpha$$

$$= \frac{N_o}{16\pi}\left[16\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right) - \left(\frac{\pi}{2}+5\right)\left(\frac{\alpha}{2}+\frac{1}{2}\sin\alpha\right)\cos\frac{\beta}{2}\,\cdot\right.$$

$$- \left(\frac{\pi}{2}+5\right)\sin^2\frac{\alpha}{2}\sin\frac{\beta}{2} + \frac{1}{2}\left(\frac{\alpha^2}{2}+\alpha\sin\alpha+\cos\alpha\right)\cos\frac{\beta}{2}$$

$$+ \frac{1}{2}\left(\sin\alpha - \alpha\cos\alpha\right)\sin\frac{\beta}{2} + \left(\frac{\pi}{2}-5\right)\sin^2\frac{\alpha}{2}\cos\frac{\beta}{2} + \left(\frac{\pi}{2}-5\right)\left(\frac{\alpha}{2}-\frac{1}{2}\sin\alpha\right).$$

$$\left.\cdot\sin\frac{\beta}{2} - \frac{1}{2}\left(\sin\alpha - \alpha\cos\alpha\right)\cos\frac{\beta}{2} - \frac{1}{2}\left(\frac{\alpha^2}{2} - \alpha\sin\alpha - \cos\alpha\right)\sin\frac{\beta}{2}\right].$$

Then

$$F^{(4)}\ (\beta,\ \beta) = \frac{N_o}{16\pi}\left[-\left(\frac{\pi}{2}+5\right)\left(\frac{\beta}{2}+\frac{1}{2}\sin\beta\right)\cos\frac{\beta}{2} - \left(\frac{\pi}{2}+5\right)\sin^2\frac{\beta}{2}\sin\frac{\beta}{2}\right.$$

$$+ \frac{1}{2}\left(\frac{\beta^2}{2}+\beta\sin\beta+\cos\beta\right)\cos\frac{\beta}{2} + \frac{1}{2}\left(\sin\beta - \beta\cos\beta\right)\sin\frac{\beta}{2}$$

$$+ \left(\frac{\pi}{2}-5\right)\sin^2\frac{\beta}{2}\cos\frac{\beta}{2} + \left(\frac{\pi}{2}-5\right)\left(\frac{\beta}{2}-\frac{1}{2}\sin\beta\right)\sin\frac{\beta}{2} - \frac{1}{2}\left(\sin\beta - \beta\cos\beta\right)\cos\frac{\beta}{2}$$

$$\left.- \frac{1}{2}\left(\frac{\beta^2}{2} - \beta\sin\beta - \cos\beta\right)\sin\frac{\beta}{2}\right]$$

$$= \frac{N_o}{16\pi}\ \left\{\cos\frac{\beta}{2}\left[-\frac{\beta}{2}\left(\frac{\pi}{2}+5\right) + \frac{1}{2}\left(\frac{\beta^2}{2}+\beta\sin\beta+\cos\beta\right) - \right.\right.$$

$$\left.-\frac{1}{2}\left(\sin\beta - \beta\cos\beta\right)\right] + \sin\frac{\beta}{2}\ \left[-\left(\frac{\pi}{2}+5\right) + \frac{1}{2}\left(\sin\beta - \beta\cos\beta\right)\right.$$

$$+\left(\frac{\pi}{2} - 5\right)\frac{\beta}{2} - \frac{1}{2}\left(\frac{\beta^2}{2} - \beta \sin \beta - \cos \beta\right)\Bigg]\Bigg\}$$

$$= \frac{N_o}{16\pi} \left\{\cos \frac{\beta}{2}\left[- \frac{\beta}{2}\left(\frac{\pi}{2} + 5\right) + \frac{1}{2}\left(\frac{\beta^2}{2} + 2\beta \sin \frac{\beta}{2} \cos \frac{\beta}{2} + 1 - 2 \sin^2 \frac{\beta}{2}\right)\right.\right.$$

$$+ \frac{1}{2}\left(2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} - \beta + 2\beta \sin^2 \frac{\beta}{2}\right)\Bigg] + \sin \frac{\beta}{2}\left[- \left(\frac{\pi}{2} + 5\right)\right.$$

$$+ \frac{1}{2}\left(2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} - 2\beta \cos^2 \frac{\beta}{2} + \beta\right) + \left(\frac{\pi}{2} - 5\right)\frac{\beta}{2}$$

$$- \frac{1}{2}\left(\frac{\beta^2}{2} - 2\beta \sin \frac{\beta}{2} \cos \frac{\beta}{2} - 2 \cos^2 \frac{\beta}{2} + 1\right)\Bigg]\Bigg\}$$

$$= \frac{N_o}{16\pi} \left\{\cos \frac{\beta}{2}\left[- \frac{\beta}{2}\left(\frac{\pi}{2} + 5\right) + \frac{1}{2}\left(\frac{\beta^2}{2} + 1\right) + \frac{\beta}{2}\right] + \sin \frac{\beta}{2}\left[- \left(\frac{\pi}{2} + 5\right) + \frac{\beta}{2}\right.\right.$$

$$+ \left(\frac{\pi}{2} - 5\right)\frac{\beta}{2} - \frac{1}{2}\left(\frac{\beta^2}{2} + 1\right)\Bigg]\Bigg\}$$

$$F^{(4)}(\beta, \beta) = \frac{N_o}{16\pi}\left\{\left[- \frac{\beta}{2}\left(\frac{\pi}{2} + 4 - \frac{\beta}{2}\right) + \frac{1}{2}\right] \cos \frac{\beta}{2} + \left[\frac{\beta}{2}\left(\frac{\pi}{2} - 4 - \frac{\beta}{2}\right) - \frac{\pi}{2} - \frac{11}{2}\right] \sin \frac{\beta}{2}\right\}$$

$$F^{(4)}(0, \beta) = \frac{N_o}{16\pi}\left(- 16 \sin \frac{\beta}{2} + \frac{1}{2} \cos \frac{\beta}{2} + \frac{1}{2} \sin \frac{\beta}{2}\right)$$

$$F^{(4)}(\pi, \beta) = \frac{N_o}{16\pi}\left[16 \cos \frac{\beta}{2} - \left(\frac{\pi}{2} + 5\right)\left(\frac{\pi}{2}\right)\cos \frac{\beta}{2} - \left(\frac{\pi}{2} + 5\right) \sin \frac{\beta}{2} + \frac{1}{2}\left(\frac{\pi^2}{2} - 1\right) \cos \frac{\beta}{2}\right.$$

$$+ \frac{1}{2} (\pi) \sin \frac{\pi}{2} + \left(\frac{\pi}{2} - 5\right) \cos \frac{\beta}{2} + \left(\frac{\pi}{2} - 5\right)\left(\frac{\pi}{2}\right) \sin \frac{\beta}{2} - \frac{1}{2} (\pi) \cos \frac{\beta}{2}$$

$$- \frac{1}{2}\left(\frac{\pi^2}{2} + 1\right) \sin \frac{\beta}{2}\Bigg].$$

$$F^{(4)}(\pi, \beta) = \frac{N_o}{16\pi}\left[\left(10\tfrac{1}{2} - \frac{5\pi}{2}\right) \cos \frac{\beta}{2} + \left(- 5\tfrac{1}{2} - \frac{5\pi}{2}\right) \sin \frac{\beta}{2}\right].$$

Then, using equations (8) and (14):

$$N_\beta^{(4)} = 2F^{(4)}(\beta, \beta) - F^{(4)}(0, \beta) - F^{(4)}(\pi, \beta) + N_\beta^{(3)}$$

$$= \frac{N_o}{16\pi}\Bigg[\left(- \frac{\pi\beta}{2} - 4\beta + \frac{\beta^2}{2} + 1 - \frac{1}{2} - 10\tfrac{1}{2} + \frac{5\pi}{2}\right) \cos \frac{\beta}{2}$$

$$+ \left(\frac{\pi\beta}{2} - 4\beta - \frac{\beta^2}{2} - \pi - 11 + 15\tfrac{1}{2} + 5\tfrac{1}{2} + \frac{5\pi}{2}\right) \sin \frac{\beta}{2} + 8\pi - 16\beta$$

$$+ 4\left(\frac{\pi}{2} - \beta + 7\right) \sin \frac{\beta}{2} + 4\left(\frac{\pi}{2} - \beta - 7\right) \cos \frac{\beta}{2}\Bigg].$$

$$N_\beta^{(4)} = \frac{N_o}{16\pi}\Bigg[8\pi - 16\beta + \left(- 38 + \frac{9\pi}{2} - 8\beta - \frac{\pi\beta}{2} + \frac{\beta^2}{2}\right) \cos \frac{\beta}{2}$$

$$+ \left(38 + \frac{7\pi}{2} - 8\beta + \frac{\pi\beta}{2} - \frac{\beta^2}{2}\right) \sin \frac{\beta}{2}\Bigg]. \tag{16}$$

From equation (9):

$$N_\alpha^{(4)} = - \frac{d}{d\beta} N^{(4)}\Big]_{\beta=\alpha}$$

$$= - \frac{N_o}{16\pi}\left[- 16 + \left(- 8 - \frac{\pi}{2} + \beta\right) \cos \frac{\beta}{2} + \left(- \frac{1}{2}\right)\left(- 38 + \frac{9\pi}{2} - 8\beta - \frac{\pi\beta}{2} + \frac{\beta^2}{2}\right)\sin \frac{\beta}{2}\right.$$

$$\left. + \left(- 8 + \frac{\pi}{2} - \beta\right) \sin \frac{\beta}{2} + \frac{1}{2}\left(38 + \frac{7\pi}{2} - 8\beta + \frac{\pi\beta}{2} - \frac{\beta^2}{2}\right) \cos \frac{\beta}{2}\right]_{\beta=\alpha}$$

$$N_\alpha^{(4)} = \frac{N_o}{32\pi}\left[32 - \left(22 + \frac{5\pi}{2} - 6\alpha + \frac{\pi\alpha}{2} - \frac{\alpha^2}{2}\right) \cos \frac{\alpha}{2} - \right.$$

$$\left. - \left(22 - \frac{7\pi}{2} + 6\alpha + \frac{\pi\alpha}{2} - \frac{\alpha^2}{2}\right) \sin \frac{\alpha}{2}\right]. \tag{17}$$

4. Fifth Collision

From equations (7) and (17):

$$F^{(5)}(\alpha, \beta) = \frac{N_o}{64\pi} \int \left[32 - \left(22 + \frac{5\pi}{2} - 6\alpha + \frac{\pi\alpha}{2} - \frac{\alpha^2}{2}\right) \cos \frac{\alpha}{2}\right.$$

$$\left. - \left(22 - \frac{7\pi}{2} + 6\alpha + \frac{\pi\alpha}{2} - \frac{\alpha^2}{2}\right) \sin \frac{\alpha}{2}\right] \left(\cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2}\right) d\alpha.$$

$$F^{(5)}(\alpha, \beta) = \frac{N_o}{64\pi} \left\{\cos \frac{\beta}{2} \left[64 \sin \frac{\alpha}{2} - \left(22 + \frac{5\pi}{2}\right)\left(\frac{\alpha}{2} + \frac{1}{2} \sin \alpha\right)\right.\right.$$

$$+ \left(6 - \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\alpha^2}{2} + \alpha \sin \alpha + \cos \alpha\right) + \frac{1}{2}\left(\frac{\alpha^3}{6} + (\frac{\alpha^2}{2} - 1) \sin \alpha + \alpha \cos \alpha\right)$$

$$+ \left(\frac{7\pi}{2} - 22\right) \sin^2 \frac{\alpha}{2} - \left(6 + \frac{\pi}{2}\right)\left(\frac{1}{2}\right) (\sin \alpha - \alpha \cos \alpha)$$

$$+ \frac{1}{2} (\alpha \sin \alpha + \cos \alpha - \frac{1}{2} \alpha^2 \cos \alpha)\Big] + \sin \frac{\beta}{2} \Big[- 64 \cos \frac{\alpha}{2} - \left(22 + \frac{5\pi}{2}\right) \sin^2 \frac{\alpha}{2}$$

$$+ \left(6 - \frac{\pi}{2}\right)\left(\frac{1}{2}\right) (\sin \alpha - \alpha \cos \alpha) + \frac{1}{2}\left(\alpha \sin \alpha + \cos \alpha - \frac{\alpha^2}{2} \cos \alpha\right)$$

$$+ \left(\frac{7\pi}{2} - 22\right)\left(\frac{\alpha}{2} - \frac{1}{2} \sin \alpha\right) - \left(6 + \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\alpha^2}{2} - \alpha \sin \alpha - \cos \alpha\right)$$

$$+ \frac{1}{2}\left(\frac{\alpha^3}{6} - (\frac{\alpha^2}{2} - 1) \sin \alpha - \alpha \cos \alpha\right)\Big]\Big\} .$$

Then,

$$F^{(5)}(\beta, \beta) = \frac{N_o}{64\pi} \Big\{\cos \frac{\beta}{2} \Big[64 \sin \frac{\beta}{2} - \left(22 + \frac{5\pi}{2}\right)\left(\frac{\beta}{2} + \sin \frac{\beta}{2} \cos \frac{\beta}{2}\right)$$

$$+ \left(6 - \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\beta^2}{2} + 2\beta \sin \frac{\beta}{2} \cos \frac{\beta}{2} + 1 - 2 \sin^2 \frac{\beta}{2}\right)$$

$$+ \frac{1}{2}\left(\frac{\beta^3}{6} + (\frac{\beta^2}{2} - 1)\ 2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} + \beta - 2\beta \sin^2 \frac{\beta}{2}\right) + \left(\frac{7\pi}{2} - 22\right) \sin^2 \frac{\beta}{2}$$

$$- \left(6 + \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} - \beta + 2\beta \sin^2 \frac{\beta}{2}\right) + \frac{1}{2}\left(2\beta \sin \frac{\beta}{2} \cos \frac{\beta}{2} + 1\right)$$

$$- 2 \sin^2 \frac{\beta}{2} - \frac{\beta^2}{2} + \beta^2 \sin^2 \frac{\beta}{2}\Big] + \sin \frac{\beta}{2} \Big[- 64 \cos \frac{\beta}{2} - \left(22 + \frac{5\pi}{2}\right)$$

$$+ \left(22 + \frac{5\pi}{2}\right)\cos^2 \frac{\beta}{2} + \left(6 - \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(2 \sin \frac{\beta}{2} \cos \frac{\beta}{2} - 2\beta \cos^2 \frac{\beta}{2} + \beta\right)$$

$$+ \frac{1}{2}\left(2\beta \sin\frac{\beta}{2} \cos\frac{\beta}{2} + 2\cos^2\frac{\beta}{2} - 1 - \beta^2 \cos^2\frac{\beta}{2} + \frac{\beta^2}{2}\right)$$

$$+ \left(\frac{7\pi}{2} - 22\right)\left(\frac{\beta}{2} - \sin\frac{\beta}{2}\cos\frac{\beta}{2}\right) - \left(6 + \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\beta^2}{2} - 2\beta \sin\frac{\beta}{2}\cos\frac{\beta}{2}\right.$$

$$\left. - 2\cos^2\frac{\beta}{2} + 1\right) + \frac{1}{2}\left(\frac{\beta^3}{6} - \left(\frac{\beta^2}{2-1}\right) \text{in}\,\frac{\beta}{2}\cos\frac{\beta}{2} - 2\beta\cos^2\frac{\beta}{2} + \beta\right)\Bigg]\Bigg\}$$

$$= \frac{N_o}{64\pi}\left\{\cos\frac{\beta}{2}\left[-\left(22 + \frac{5\pi}{2}\right)\frac{\beta}{2} + \left(6 - \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\beta^2}{2} + 1\right) + \frac{1}{2}\left(\frac{\beta^3}{6} + \beta\right)\right.\right.$$

$$\left. - \left(6 + \frac{\pi}{2}\right)\left(\frac{1}{2}\right)(-\beta) + \frac{1}{2}\left(1 - \frac{\beta^2}{2}\right)\right] + \sin\frac{\beta}{2}\left[-\left(22 + \frac{5\pi}{2}\right)\right.$$

$$+ \left(6 - \frac{\pi}{2}\right)\left(\frac{1}{2}\right)(\beta) + \frac{1}{2}\left(-1 + \frac{\beta^2}{2}\right) + \left(\frac{7\pi}{2} - 22\right)\frac{\beta}{2} - \left(6 + \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\beta^2}{2} + 1\right)$$

$$\left.\left. + \frac{1}{2}\left(\frac{\beta^3}{6} + \beta\right)\right]\right\}.$$

$$F^{(5)}(\beta, \beta) = \frac{N_o}{64\pi}\left[\left(\frac{7}{2} - \frac{\pi}{4} - \frac{15\beta}{2} - \pi\beta + \frac{5\beta^2}{4} - \frac{\pi\beta^2}{8} + \frac{\beta^3}{12}\right)\cos\frac{\beta}{2}\right.$$

$$\left. + \left(-25\frac{1}{2} - \frac{11\pi}{4} - \frac{15\beta}{2} + \frac{3\pi\beta}{2} - \frac{5\beta^2}{4} - \frac{\pi\beta^2}{8} + \frac{\beta^3}{12}\right)\sin\frac{\beta}{2}\right]$$

$$F^{(5)}(0, \beta) = \frac{N_o}{64\pi}\left\{\cos\frac{\beta}{2}\left[\left(6 - \frac{\pi}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{2}\right] + \sin\frac{\beta}{2}\left[-64 + \frac{1}{2} + \right.\right.$$

$$\left.\left. + \left(6 + \frac{\pi}{2}\right)\left(\frac{1}{2}\right)(1)\right]\right\}.$$

$$F^{(5)}(0, \beta) = \frac{N_o}{64\pi}\left[\left(\frac{7}{2} - \frac{\pi}{4}\right)\cos\frac{\beta}{2} + \left(-\,60\frac{1}{2} + \frac{\pi}{4}\right)\sin\frac{\beta}{2}\right].$$

$$F^{(5)}(\pi, \beta) = \frac{N_o}{64\pi}\left\{\cos\frac{\beta}{2}\left[64 - \left(22 + \frac{5\pi}{2}\right)\left(\frac{\pi}{2}\right) + \left(6 - \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi^2}{2} - 1\right)\right.\right.$$

$$\left. + \frac{1}{2}\left(\frac{\pi^3}{6} - \pi\right) + \left(\frac{7\pi}{2} - 22\right) - \left(6 + \frac{\pi}{2}\right)\left(\frac{1}{2}\right)(\pi) + \frac{1}{2}\left(-\,1 + \frac{\pi^2}{2}\right)\right]$$

$$+ \sin\frac{\beta}{2}\left[-\left(22 + \frac{5\pi}{2}\right) + \left(6 - \frac{\pi}{2}\right)\left(\frac{1}{2}\right)(\pi) + \frac{1}{2}\left(-\,1 + \frac{\pi^2}{2}\right) + \left(\frac{7\pi}{2} - 22\right)\left(\frac{\pi}{2}\right)\right.$$

$$\left.\left. - \left(6 + \frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi^2}{2} + 1\right) + \frac{1}{2}\left(\frac{\pi^3}{6} + \pi\right)\right]\right\}.$$

$$F^{(5)}(\pi, \beta) = \frac{N_o}{64\pi}\left[\cos\frac{\beta}{2}\left(38\frac{1}{2} - 10\frac{3}{4}\pi + \frac{\pi^2}{4} + \frac{\pi^3}{24}\right)\right.$$

$$\left. + \sin\frac{\beta}{2}\left(-\,25\frac{1}{2} - 10\frac{1}{4}\pi + \frac{\pi^2}{4} - \frac{\pi^3}{24}\right)\right].$$

Then, using equations (8) and (16):

$$N_\beta^{(5)} = 2F^{(5)}(\beta, \beta) - F^{(5)}(0, \beta) - F^{(5)}(\pi, \beta) + N_\beta^{(4)}$$

$$= \frac{N_o}{64\pi}\left[\cos\frac{\beta}{2}\left(7 - \frac{\pi}{2} - 15\beta - 2\pi\beta + \frac{5\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6} - \frac{7}{2} + \frac{\pi}{4} - 38\frac{1}{2} + 10\frac{3}{4}\,\pi\right.\right.$$

$$- \frac{\pi^2}{4} + \frac{\pi^3}{24} - 152 + 18\pi - 32\beta - 2\pi\beta + 2\beta^2\Big) + \sin\frac{\beta}{2}\Big(- 51 - \frac{11\pi}{2} - 15\beta$$

$$+3\pi\beta - \frac{5\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6} + 60\frac{1}{2} - \frac{\pi}{4} + 25\frac{1}{2} + 10\frac{1}{4}\pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} + 152 + 14\pi$$

$$- 32\beta + 2\pi\beta - 2\beta^2\Big) + 32\pi - 64\beta\Big].$$

$$N_\beta^{(5)} = \frac{N_o}{64\pi}\Big[32\pi - 64\beta + \Big(- 187 + 28\frac{1}{2}\pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta - 4\pi\beta + \frac{9\beta^2}{2}$$

$$- \frac{\pi\beta^2}{4} + \frac{\beta^3}{6}\Big)\cos\frac{\beta}{2} + \Big(187 + 18\frac{1}{2}\pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta + 5\pi\beta - \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4}$$

$$+ \frac{\beta^3}{6}\Big)\sin\frac{\beta}{2}\Big] \tag{18}$$

From equation (9):

$$N_\beta^{(5)} = - \frac{d}{d\beta} N_\beta^{(5)}\Big]_{\beta=\alpha} = - \frac{N_o}{64\pi}\Big[- 64 + \Big(- \frac{1}{2}\Big)\Big(- 187 + 28\frac{1}{2}\pi - \frac{\pi^2}{4} + \frac{\pi^3}{24}$$

$$- 47\beta - 4\pi\beta + \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6}\Big)\sin\frac{\beta}{2} + \Big(- 47 - 4\pi + 9\beta - \frac{\pi\beta}{2} + \frac{\beta^2}{2}\Big)\cos\frac{\beta}{2}$$

$$+ \frac{1}{2}\Big(187 + 18\frac{1}{2}\pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta + 5\pi\beta - \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6}\Big)\cos\frac{\beta}{2}$$

$$+\left(-\ 47 + 5\pi - 9\beta - \frac{\pi\beta}{2} + \frac{\beta^2}{2}\right)\sin\frac{\beta}{2}\Bigg]_{\beta=\alpha}$$

$$N_\alpha^{(5)} = \frac{N_o}{128\pi}\Bigg[128 + \left(-\ 93 - 10\tfrac{1}{2}\,\pi + \frac{\pi^2}{4} - \frac{\pi^3}{24} + 29\alpha - 4\pi\alpha + \frac{7\alpha^2}{2} - \frac{\pi\alpha^2}{4} - \frac{\alpha^3}{6}\right)\cos\frac{\alpha}{2}$$

$$+\left(-\ 93 + 18\tfrac{1}{2}\,\pi - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 29\alpha - 3\pi\alpha + \frac{7\alpha^2}{2} - \frac{\pi\alpha^2}{4} + \frac{\alpha^3}{6}\right)\sin\frac{\alpha}{2}\Bigg]. \qquad (19)$$

C. Case II: Incident Distribution Uniform across the Y-Axis

1. Distribution at Second Collision

For an incident distribution of molecules uniform across the y-axis:

$$N_\alpha \equiv \frac{dN}{d\alpha} \neq \text{constant}$$

but

$$\frac{dN}{dy} = \text{constant} = k.$$

Now

$$N_\alpha \equiv \frac{dN}{d\alpha} = \frac{dN}{dy} \cdot \frac{dy}{d\alpha}\ ,$$

and from Figure 1 it is clear that $y = -\cos\alpha$ so that

$$\frac{dy}{d\alpha} = \sin\alpha.$$

Therefore, $N_\alpha = k \sin \alpha$, but

$$N_o = \int_o^{\pi} k \sin \alpha \, d\alpha = 2k$$

so that

$$N_\alpha = \frac{1}{2} N_o \sin \alpha \; . \qquad (20)$$

From equation (5):

$$N_\beta^{(1)} = \int_\beta^{\pi/2} N_\alpha^{(1)} \, d\alpha = \int_\beta^{\pi/2} \frac{1}{2} N_o \sin \alpha \, d\alpha$$

$$= \frac{1}{2} N_o (- \cos \frac{\pi}{2} + \cos \beta)$$

$$N_\beta^{(1)} = \frac{1}{2} N_o \cos \beta \qquad (21)$$

From equations (7) and (20):

$$F^{(2)}(\alpha, \beta) = \frac{1}{2} \int \frac{1}{2} N_o \sin \alpha \left(\cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha$$

$$= \frac{N_o}{2} \int \left(\cos \frac{\beta}{2} \sin \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} + \sin \frac{\beta}{2} \sin^2 \frac{\alpha}{2} \cos \frac{\alpha}{2} \right) d\alpha$$

$$F^{(2)}(\alpha, \beta) = \frac{N_o}{3} \left(\sin \frac{\beta}{2} \sin^3 \frac{\alpha}{2} - \cos \frac{\beta}{2} \cos^3 \frac{\alpha}{2} \right).$$

Then

$$F^{(2)}(\beta,\ \beta) = \frac{N_o}{3}\left(\sin^4 \frac{\beta}{2} - \cos^4 \frac{\beta}{2}\right)$$

$$= \frac{N_o}{3}\left(\sin^2 \frac{\beta}{2} - \cos^2 \frac{\beta}{2}\right)\left(\sin^2 \frac{\beta}{2} + \cos^2 \frac{\beta}{2}\right).$$

$$F^{(2)}(\beta,\ \beta) = \frac{N_o}{3}\,(-\cos\beta)$$

$$F^{(2)}(0,\ \beta) = \frac{N_o}{3}\,(-\cos\frac{\beta}{2})$$

$$F^{(2)}(\pi,\ \beta) = \frac{N_o}{3}\,(\sin\frac{\beta}{2}).$$

Then, using equations (8) and (21):

$$N_\beta^{(2)} = 2F^{(2)}(\beta,\ \beta) - F^{(2)}(0,\ \beta) - F^{(2)}(\pi,\ \beta) + N_\beta^{(1)},$$

$$= \frac{N_o}{3}\left(-\ 2\cos\beta + \cos\frac{\beta}{2} - \sin\frac{\beta}{2} + \frac{3}{2}\cos\beta\right),$$

$$N_\beta^{(2)} = \frac{N_o}{3}\left(\cos\frac{\beta}{2} - \sin\frac{\beta}{2} - \frac{1}{2}\cos\beta\right). \qquad (22)$$

From equation (9):

$$N_\alpha^{(2)} = -\left.\frac{d}{d\beta}N_\beta^{(2)}\right]_{\beta=\alpha} = -\frac{N_o}{3}\left[-\frac{1}{2}\sin\frac{\beta}{2} - \frac{1}{2}\cos\frac{\beta}{2} + \frac{1}{2}\sin\beta\right]_{\beta=\alpha}$$

$$N_\alpha^{(2)} = \frac{N_o}{6}\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} - \sin\alpha\right). \qquad (23)$$

2. Third Collision

From equations (7) and (23):

$$F^{(3)}(\alpha,\ \beta) = \frac{N_o}{12} \int \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} - \sin\alpha\right)\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) d\alpha$$

$$= \frac{N_o}{12}\left[\cos\frac{\beta}{2}\left(\frac{\alpha}{2} + \frac{1}{2}\sin\alpha + \sin^2\frac{\alpha}{2} + \frac{4}{3}\cos^3\frac{\alpha}{2}\right)\right.$$

$$\left. + \sin\frac{\beta}{2}\left(\sin^2\frac{\alpha}{2} + \frac{\alpha}{2} - \frac{1}{2}\sin\alpha - \frac{4}{3}\sin^3\frac{\alpha}{2}\right)\right]$$

Then

$$F^{(3)}(\beta,\ \beta) = \frac{N_o}{12}\left[\cos\frac{\beta}{2}\left(\frac{\beta}{2} + \frac{1}{2}\sin\beta + \sin^2\frac{\beta}{2} + \frac{4}{3}\cos^3\frac{\beta}{2}\right)\right.$$

$$\left. + \sin\frac{\beta}{2}\left(\sin^2\frac{\beta}{2} + \frac{\beta}{2} - \frac{1}{2}\sin\beta - \frac{4}{3}\sin^3\frac{\beta}{2}\right)\right]$$

$$= \frac{N_o}{12}\left[\cos\frac{\beta}{2}\left(\frac{\beta}{2} + \sin\frac{\beta}{2}\cos\frac{\beta}{2} + \sin^2\frac{\beta}{2} + \frac{4}{3}\cos\frac{\beta}{2} - \frac{4}{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\right)\right.$$

$$\left. + \sin\frac{\beta}{2}\left(1 - \cos^2\frac{\beta}{2} + \frac{\beta}{2} - \sin\frac{\beta}{2}\cos\frac{\beta}{2} - \frac{4}{3}\sin\frac{\beta}{2} + \frac{4}{3}\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}\right)\right]$$

$$F^{(3)}(\beta,\ \beta) = \frac{N_o}{12}\left[\frac{4}{3}\cos\beta + \frac{\beta}{2}\cos\frac{\beta}{2} + \left(1 + \frac{\beta}{2}\right)\sin\frac{\beta}{2}\right]$$

$$F^{(3)}(0,\ \beta) = \frac{N_o}{12}\left[\cos\frac{\beta}{2}\cdot\left(\frac{4}{3}\right)\right]$$

$$F^{(3)}(\pi,\ \beta) = \frac{N_o}{12}\left[\cos\frac{\beta}{2}\left(\frac{\pi}{2} + 1\right) + \sin\frac{\beta}{2}\left(1 + \frac{\pi}{2} - \frac{4}{3}\right)\right]$$

From equation (22):

$$N_\beta^{(2)} = \frac{N_o}{12}\left(4 \cos \frac{\beta}{2} - 4 \sin \frac{\beta}{2} - 2 \cos \beta\right).$$

Then, using equation (8):

$$N_\beta^{(3)} = 2F^{(3)}(\beta, \beta) - F^{(3)}(0, \beta) - F^{(3)}(\pi, \beta) + N_\beta^{(2)}$$

$$= \frac{N_o}{12}\left[\frac{8}{3} \cos \beta + \beta \cos \frac{\beta}{2} + (2 + \beta) \sin \frac{\beta}{2} - \frac{4}{3} \cos \frac{\beta}{2} - (\frac{\pi}{2} + 1) \cos \frac{\beta}{2}\right.$$

$$\left. - \left(\frac{\pi}{2} - \frac{1}{3}\right) \sin \frac{\beta}{2} + 4 \cos \frac{\beta}{2} - 4 \sin \frac{\beta}{2} - 2 \cos \beta\right].$$

$$N_\beta^{(3)} = \frac{N_o}{12}\left[\frac{2}{3} \cos \beta + \left(\beta - \frac{\pi}{2} + \frac{5}{3}\right) \cos \frac{\beta}{2} + \left(\beta - \frac{\pi}{2} - \frac{5}{3}\right) \sin \frac{\beta}{2}\right]. \quad (24)$$

From equation (9):

$$N_\alpha^{(3)} = -\frac{d}{d\beta} N_\beta^{(3)}\Big]_{\beta=\alpha} = -\frac{N_o}{12}\left[-\frac{2}{3} \sin \beta + \left(\beta - \frac{\pi}{2} + \frac{5}{3}\right)\left(- \sin \frac{\beta}{2}\right)\left(\frac{1}{2}\right)\right.$$

$$\left. + \cos \frac{\beta}{2} + \left(\beta - \frac{\pi}{2} - \frac{5}{3}\right)\left(\cos \frac{\beta}{2}\right)\left(\frac{1}{2}\right) + \sin \frac{\beta}{2}\right]_{\beta=\alpha}$$

$$= -\frac{N_o}{12}\left[-\frac{2}{3} \sin \beta + \left(-\frac{\beta}{2} + \frac{\pi}{4} + \frac{1}{6}\right) \sin \frac{\beta}{2} + \left(\frac{\beta}{2} - \frac{\pi}{4} + \frac{1}{6}\right) \cos \frac{\beta}{2}\right]_{\beta=\alpha}$$

$$N_\alpha^{(3)} = \frac{N_o}{12}\left[\left(-\frac{1}{6} - \frac{\pi}{4} + \frac{\alpha}{2}\right) \sin \frac{\alpha}{2} + \left(-\frac{1}{6} + \frac{\pi}{4} - \frac{\alpha}{2}\right) \cos \frac{\alpha}{2} + \frac{2}{3} \sin \alpha\right]. \quad (25)$$

3. Fourth Collision

From equations (7) and (25):

$$F^{(4)}(\alpha,\ \beta) = \frac{N_o}{24} \int \left[\left(-\frac{1}{6} - \frac{\pi}{4} + \frac{\alpha}{2}\right) \sin\frac{\alpha}{2} + \left(-\frac{1}{6} + \frac{\pi}{4} - \frac{\alpha}{2}\right) \cos\frac{\alpha}{2} + \frac{2}{3} \sin\alpha\right.$$

$$\left.\cdot \left(\cos\frac{\alpha}{2} \cos\frac{\beta}{2} + \sin\frac{\alpha}{2} \sin\frac{\beta}{2}\right) d\alpha.\right.$$

$$= \frac{N_o}{24} \left\{\cos\frac{\beta}{2} \left[\left(-\frac{1}{6} - \frac{\pi}{4}\right) \sin^2\frac{\alpha}{2} + \frac{1}{4} \sin\alpha - \frac{1}{4}\alpha \cos\alpha + \left(-\frac{1}{6} + \frac{\pi}{4}\right)\right.\right.$$

$$\left.\cdot \left(\frac{\alpha}{2} + \frac{1}{2}\sin\alpha\right) - \frac{1}{2}\left(\frac{\alpha^2}{4} + \frac{\alpha\sin\alpha}{2} + \frac{\cos\alpha}{2}\right) - \frac{8}{9}\cos^3\frac{\alpha}{2}\right]$$

$$+ \sin\frac{\beta}{2}\left[\left(-\frac{1}{6} - \frac{\pi}{4}\right)\left(\frac{\alpha}{2} - \frac{1}{2}\sin\alpha\right) + \frac{1}{2}\left(\frac{\alpha^2}{4} - \frac{\alpha\sin\alpha}{2} - \frac{\cos\alpha}{2}\right)\right.$$

$$\left.\left. + \left(-\frac{1}{6} + \frac{\pi}{4}\right)\sin^2\frac{\alpha}{2} - \frac{1}{4}\sin\alpha + \frac{1}{4}\alpha\cos\alpha + \frac{8}{9}\sin^3\frac{\alpha}{2}\right]\right\}$$

$$F^{(4)}(\alpha,\ \beta) = \frac{N_o}{24}\left\{\cos\frac{\beta}{2}\left[\left(-\frac{1}{6} - \frac{\pi}{4}\right)\sin^2\frac{\alpha}{2} + \left(\frac{1}{6} + \frac{\pi}{8} - \frac{\alpha}{4}\right)\sin\alpha + \right.\right.$$

$$\left.\left(-\frac{1}{4} - \frac{\alpha}{4}\right)\cos\alpha + \left(-\frac{1}{12} + \frac{\pi}{8}\right)\alpha - \frac{\alpha^2}{8} - \frac{8}{9}\cos^3\frac{\alpha}{2}\right]$$

$$+ \sin\frac{\beta}{2}\left[\left(-\frac{1}{6} + \frac{\pi}{4}\right)\sin^2\frac{\alpha}{2} + \left(-\frac{1}{6} + \frac{\pi}{8} - \frac{\alpha}{4}\right)\sin\alpha + \left(-\frac{1}{4} + \frac{\alpha}{4}\right)\cos\alpha\right.$$

$$\left.\left. + \left(-\frac{1}{12} - \frac{\pi}{8}\right)\alpha + \frac{\alpha^2}{8} + \frac{8}{9}\sin^3\frac{\alpha}{2}\right]\right\}$$

$$F^{(4)}(\beta, \beta) = \frac{N_o}{24}\left\{\cos\frac{\beta}{2}\left[\left(-\frac{1}{6}-\frac{\pi}{4}\right)\sin^2\frac{\beta}{2} + \left(\frac{1}{6}+\frac{\pi}{8}-\frac{\beta}{4}\right)\sin\beta\right.\right.$$

$$+\left(-\frac{1}{4}-\frac{\beta}{4}\right)\cos\beta + \left(-\frac{1}{12}+\frac{\pi}{8}\right)\beta - \frac{\beta^2}{8} - \frac{8}{9}\cos^3\frac{\beta}{2}\Bigg]$$

$$+\sin\frac{\beta}{2}\left[\left(-\frac{1}{6}+\frac{\pi}{4}\right)\sin^2\frac{\beta}{2} + \left(-\frac{1}{6}+\frac{\pi}{8}-\frac{\beta}{4}\right)\sin\beta + \left(-\frac{1}{4}+\frac{\beta}{4}\right)\cos\beta\right.$$

$$\left.\left.+\left(-\frac{1}{12}-\frac{\pi}{8}\right)\beta + \frac{\beta^2}{8} + \frac{8}{9}\sin^3\frac{\beta}{2}\right]\right\}$$

$$= \frac{N_o}{24}\left\{\left(-\frac{1}{6}-\frac{\pi}{4}\right)\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} + \left(\frac{1}{3}+\frac{\pi}{4}-\frac{\beta}{2}\right)\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\right.$$

$$+\left(-\frac{1}{4}-\frac{\beta}{4}\right)\cos\frac{\beta}{2} - \left(-\frac{1}{2}-\frac{\beta}{2}\right)\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} + \left[\left(-\frac{1}{12}+\frac{\pi}{8}\right)\beta - \frac{\beta^2}{8}\right]\cos\frac{\beta}{2}$$

$$-\frac{8}{9}\cos^2\frac{\beta}{2} + \frac{8}{9}\cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2} + \left(-\frac{1}{6}+\frac{\pi}{4}\right)\sin\frac{\beta}{2} - \left(-\frac{1}{6}+\frac{\pi}{4}\right)\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}$$

$$+\left(-\frac{1}{3}+\frac{\pi}{4}-\frac{\beta}{2}\right)\sin^2\frac{\beta}{2}\cos\frac{\beta}{2} + 2\left(-\frac{1}{4}+\frac{\beta}{4}\right)\sin\frac{\beta}{2}\cos^2\frac{\beta}{2} - \left(-\frac{1}{4}+\frac{\beta}{4}\right)\sin\frac{\beta}{2}$$

$$\left.+\left[\left(-\frac{1}{12}-\frac{\pi}{8}\right)\beta + \frac{\beta^2}{8}\right]\sin\frac{\beta}{2} + \frac{8}{9}\sin^2\frac{\beta}{2} - \frac{8}{9}\sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2}\right\}.$$

$$F^{(4)}(\beta,\ \beta) = \frac{N_o}{24}\left[\cos\frac{\beta}{2}\left(-\frac{\beta}{3} + \frac{\pi\beta}{8} - \frac{1}{4} - \frac{\beta^2}{8}\right) + \sin\frac{\beta}{2}\left(\frac{1}{12} + \frac{\pi}{4} - \frac{\beta}{3} - \frac{\pi\beta}{8} + \frac{\beta^2}{8}\right)\right.$$

$$\left. - \frac{8}{9}\cos\beta\right].$$

$$F^{(4)}(0,\ \beta) = \frac{N_o}{24}\left[\cos\frac{\beta}{2}\left(-\frac{1}{4} - \frac{8}{9}\right) + \sin\frac{\beta}{2}\left(-\frac{1}{4}\right)\right].$$

$$F^{(4)}(\pi,\ \beta) = \frac{N_o}{24}\left\{\cos\frac{\beta}{2}\left[\left(-\frac{1}{6} - \frac{\pi}{4}\right) - \left(-\frac{1}{4} - \frac{\pi}{4}\right) + \left(-\frac{1}{12} + \frac{\pi}{8}\right)\pi - \frac{\pi^2}{8}\right] +\right.$$

$$\left.\sin\frac{\beta}{2}\left[\left(-\frac{1}{6} + \frac{\pi}{4}\right) + \left(\frac{1}{4} - \frac{\pi}{4}\right) + \left(-\frac{1}{12} - \frac{\pi}{8}\right)\pi + \frac{\pi^2}{8} + \frac{8}{9}\right]\right\}.$$

$$F^{(4)}(\pi,\ \beta) = \frac{N_o}{24}\left[\cos\frac{\beta}{2}\left(\frac{1}{12} - \frac{\pi}{12}\right) + \sin\frac{\beta}{2}\left(\frac{1}{12} - \frac{\pi}{12}\right) + \frac{8}{9}\sin\frac{\beta}{2}\right].$$

From equation (24):

$$N_\beta^{(3)} = \frac{N_o}{24}\left[\left(-\frac{10}{3} - \pi + 2\beta\right)\sin\frac{\beta}{2} + \left(\frac{10}{3} - \pi + 2\beta\right)\cos\frac{\beta}{2} + \frac{4}{3}\cos\beta\right].$$

Then, from equation (8):

$$N_\beta^{(4)} = 2F^{(4)}(\beta,\ \beta) - F^{(4)}(0,\ \beta) - F^{(4)}(\pi,\ \beta) + N_\beta^{(3)}$$

$$= \frac{N_o}{24}\left[\cos\frac{\beta}{2}\left(-\frac{2\beta}{3} + \frac{\pi\beta}{4} - \frac{1}{2} - \frac{\beta^2}{4} + \frac{1}{4} + \frac{8}{9} - \frac{1}{12} + \frac{\pi}{12} + \frac{10}{3} - \pi + 2\beta\right)\right.$$

$$+ \sin\frac{\beta}{2}\left(\frac{1}{6} + \frac{\pi}{2} - \frac{2\beta}{3} - \frac{\pi\beta}{4} + \frac{\beta^2}{4} + \frac{1}{4} - \frac{1}{12} + \frac{\pi}{12} - \frac{8}{9} - \frac{10}{3} - \pi + 2\beta\right) - \frac{4}{9}\cos\beta\Bigg]$$

$$N_\beta^{(4)} = \frac{N_o}{24}\Bigg[\cos\frac{\beta}{2}\left(3\frac{8}{9} + \frac{4\beta}{3} + \frac{\pi\beta}{4} - \frac{\beta^2}{4} - \frac{11\pi}{12}\right)$$

$$+ \sin\frac{\beta}{2}\left(-3\frac{8}{9} + \frac{4\beta}{3} - \frac{\pi\beta}{4} + \frac{\beta^2}{4} - \frac{5\pi}{12}\right) - \frac{4}{9}\cos\beta\Bigg] \tag{26}$$

From equation (9):

$$N_\alpha^{(4)} = -\frac{d}{d\beta}N_\beta^{(4)}\Bigg]_{\beta=\alpha} = -\frac{N_o}{24}\Bigg[\cos\frac{\beta}{2}\left(\frac{4}{3} + \frac{\pi}{4} - \frac{2\beta}{4}\right)$$

$$+ \frac{1}{2}\cos\frac{\beta}{2}\left(-3\frac{8}{9} + \frac{4\beta}{3} + \frac{\pi\beta}{4} + \frac{\beta^2}{4} - \frac{5\pi}{12}\right)$$

$$+ \left(-\frac{1}{2}\sin\frac{\beta}{2}\right)\left(3\frac{8}{9} + \frac{4\beta}{3} + \frac{\pi\beta}{4} - \frac{\beta^2}{4} - \frac{11\pi}{12}\right)$$

$$+ \sin\frac{\beta}{2}\left(\frac{4}{3} - \frac{\pi}{4} + \frac{\beta}{2}\right) + \frac{4}{9}\sin\beta\Bigg]_{\beta=\alpha}$$

$$N_\alpha^{(4)} = \frac{N_o}{24}\Bigg[\left(\frac{11}{18} - \frac{\pi}{24} - \frac{\alpha}{6} + \frac{\pi\alpha}{8} - \frac{\alpha^2}{8}\right)\cos\frac{\alpha}{2} + \left(\frac{11}{18} - \frac{5\pi}{24} + \frac{\alpha}{6} + \frac{\pi\alpha}{8} - \frac{\alpha^2}{8}\right)\sin\frac{\alpha}{2}$$

$$- \frac{4}{9}\sin\alpha\Bigg]. \tag{27}$$

4. Fifth Collision

From equations (7) and (27):

$$F^{(5)}(\alpha, \beta) = -\frac{N_o}{48} \int \left[\left(-\frac{11}{18} + \frac{\pi}{24} + \frac{\alpha}{6} - \frac{\pi\alpha}{8} + \frac{\alpha^2}{8} \right) \cos \frac{\alpha}{2} \right.$$

$$\left. + \left(-\frac{11}{18} + \frac{5\pi}{24} - \frac{\alpha}{6} - \frac{\pi\alpha}{8} + \frac{\alpha^2}{8} \right) \sin \frac{\alpha}{2} + \frac{4}{9} \sin \alpha \right] \left(\cos \frac{\alpha}{2} \cos \frac{\beta}{2} \right.$$

$$\left. + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \right) d\alpha$$

$$= -\frac{N_o}{48} \left\{ \cos \frac{\beta}{2} \left[\left(-\frac{11}{18} + \frac{\pi}{24} \right) \left(\frac{\alpha}{2} + \frac{1}{2} \sin \alpha \right) + \left(\frac{1}{6} - \frac{\pi}{8} \right) \left(\frac{\alpha^2}{4} + \frac{\alpha \sin \alpha}{2} + \frac{\cos \alpha}{2} \right) \right. \right.$$

$$+ \frac{1}{8} \left(\frac{\alpha^3}{6} + (\frac{\alpha^2}{2} - 1) \sin \alpha + \alpha \cos \alpha \right) + \left(-\frac{11}{18} + \frac{5\pi}{24} \right) \sin^2 \frac{\alpha}{2}$$

$$- \left(\frac{1}{6} + \frac{\pi}{8} \right) \left(\frac{1}{2} \right) \left(\sin \alpha - \alpha \cos \alpha \right) + \frac{1}{8} \left(\alpha \sin \alpha + \cos \alpha - \frac{1}{2} \alpha^2 \cos \alpha \right)$$

$$\left. + \frac{4}{9} \left(-\frac{4}{3} \cos^3 \frac{\alpha}{2} \right) \right] + \sin \frac{\beta}{2} \left[\left(-\frac{11}{18} + \frac{\pi}{24} \right) \sin^2 \frac{\alpha}{2} + \left(\frac{1}{6} - \frac{\pi}{8} \right) \left(\frac{1}{2} \right) \cdot \right.$$

$$\cdot \left(\sin \alpha - \alpha \cos \alpha \right) + \frac{1}{8} \left(\alpha \sin \alpha + \cos \alpha - \frac{1}{2} \alpha^2 \cos \alpha \right)$$

$$+ \left(-\frac{11}{18} + \frac{5\pi}{24} \right) \left(\frac{\alpha}{2} - \frac{1}{2} \right) \sin \alpha - \left(\frac{1}{6} + \frac{\pi}{8} \right) \left(\frac{\alpha^2}{4} - \frac{\alpha \sin \alpha}{2} - \frac{\cos \alpha}{2} \right)$$

$$+ \frac{1}{8}\left(\frac{\alpha^3}{6} - (\frac{\alpha^2}{2} - 1) \sin\alpha - \alpha\cos\alpha\right) + \frac{4}{9}\left(\frac{4}{3}\sin^3\frac{\alpha}{2}\right)\Bigg]\Bigg\}$$

$$F^{(5)}(\alpha, \beta) = -\frac{N_o}{48}\left\{\cos\frac{\beta}{2}\left[\left(-\frac{11}{36} + \frac{\pi}{48}\right)\alpha + \left(\frac{1}{24} - \frac{\pi}{32}\right)\alpha^2\right.\right.$$

$$+ \left(-\frac{37}{72} + \frac{5\alpha}{24} - \frac{\pi\alpha}{16} - \frac{\pi}{24} + \frac{\alpha^2}{16}\right)\sin\alpha + \left(\frac{5}{24} + \frac{5\alpha}{24} + \frac{\pi\alpha}{16} - \frac{\pi}{16} - \frac{\alpha^2}{16}\right)\cos\alpha$$

$$+ \frac{\alpha^3}{48} + \left(-\frac{11}{18} + \frac{5\pi}{24}\right)\sin^2\frac{\alpha}{2} - \frac{16}{27}\cos^3\frac{\alpha}{2}\Bigg] + \sin\frac{\beta}{2}\left[\left(-\frac{11}{36} + \frac{5\pi}{48}\right)\alpha\right.$$

$$+ \left(-\frac{1}{24} - \frac{\pi}{32}\right)\alpha^2 + \left(\frac{37}{72} + \frac{5\alpha}{24} + \frac{\pi\alpha}{16} - \frac{\pi}{6} - \frac{\alpha^2}{16}\right)\sin\alpha$$

$$+ \left(\frac{5}{24} - \frac{5\alpha}{24} + \frac{\pi\alpha}{16} + \frac{\pi}{16} - \frac{\alpha^2}{16}\right)\cos\alpha + \frac{\alpha^3}{48} + \left(-\frac{11}{18} + \frac{\pi}{24}\right)\sin^2\frac{\alpha}{2}$$

$$\left. \left. + \frac{16}{27}\sin^3\frac{\alpha}{2}\right]\right\}.$$

$$F^{(5)}(\beta, \beta) = -\frac{N_o}{48}\left\{\cos\frac{\beta}{2}\left[\left(-\frac{11}{36} + \frac{\pi}{48}\right)\beta + \left(\frac{1}{24} - \frac{\pi}{32}\right)\beta^2\right.\right.$$

$$+ \left(-\frac{37}{36} + \frac{5\alpha}{12} - \frac{\pi\alpha}{8} - \frac{\pi}{12} + \frac{\beta^2}{8}\right)\sin\frac{\beta}{2}\cos\frac{\beta}{2} + \left(\frac{5}{24} + \frac{5\beta}{24} + \frac{\pi\beta}{16} - \frac{\pi}{16} - \frac{\beta^2}{16}\right)$$

$$\cdot \left(1 - 2\sin^2\frac{\beta}{2}\right) + \frac{\beta^3}{48} + \left(-\frac{11}{18} + \frac{5\pi}{24}\right)\sin^2\frac{\beta}{2} - \frac{16}{27}\cos\frac{\alpha}{2}\left(1 - \sin^2\frac{\alpha}{2}\right)\Bigg]$$

$$+ \sin\frac{\beta}{2}\left[\left(-\frac{11}{36} + \frac{5\pi}{48}\right)\beta + \left(-\frac{1}{24} - \frac{\pi}{32}\right)\beta^2 + \left(\frac{37}{36} + \frac{5\beta}{12} + \frac{\pi\beta}{8} - \frac{\pi}{3} - \frac{\beta^2}{8}\right)\right.$$

$$\cdot \sin\frac{\beta}{2}\cos\frac{\beta}{2} + \left(\frac{5}{24} - \frac{5\beta}{24} + \frac{\pi\beta}{16} + \frac{\pi}{16} - \frac{\beta^2}{16}\right)\left(2\cos^2\frac{\beta}{2} - 1\right) + \frac{\beta^3}{48}$$

$$\left. + \left(-\frac{11}{18} + \frac{\pi}{24}\right)\left(1 - \cos^2\frac{\beta}{2}\right) + \frac{16}{27}\sin\frac{\beta}{2}\left(1 - \cos^2\frac{\beta}{2}\right)\right]\Bigg\} .$$

$$F^{(5)}(\beta, \beta) = -\frac{N_o}{48}\left\{\cos\frac{\beta}{2}\left[\frac{5}{24} - \frac{\pi}{16} + \left(\frac{\pi}{12} - \frac{7}{72}\right)\beta - \left(\frac{1}{48} + \frac{\pi}{32}\right)\beta^2 + \frac{1}{48}\beta^3\right]\right.$$

$$\left. + \sin\frac{\beta}{2}\left[-\frac{59}{72} - \frac{\pi}{48} + \left(\frac{\pi}{24} - \frac{7}{72}\right)\beta + \left(\frac{1}{48} - \frac{\pi}{32}\right)\beta^2 + \frac{1}{48}\beta^3\right] - \frac{16}{27}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right)\right\}$$

$$F^{(5)}(0, \beta) = -\frac{N_o}{48}\left\{\cos\frac{\beta}{2}\left[\frac{5}{24} - \frac{\pi}{16} - \frac{16}{27}\right] + \sin\frac{\beta}{2}\left[\frac{5}{24} + \frac{\pi}{16}\right]\right\} .$$

$$F^{(5)}(\pi, \beta) = -\frac{N_o}{48}\left\{\cos\frac{\beta}{2}\left[\left(-\frac{11}{36} + \frac{\pi}{48}\right)\pi + \left(\frac{1}{24} - \frac{\pi}{32}\right)\pi^2\right.\right.$$

$$\left. - \frac{5}{24} + \frac{\pi}{16} + \frac{\pi^3}{48} - \frac{11}{18}\right] + \sin\frac{\beta}{2}\left[\left(-\frac{11}{36} + \frac{5\pi}{48}\right)\pi + \right.$$

$$+ \left(- \frac{1}{24} - \frac{\pi}{32}\right) \pi^2 - \left(\frac{5}{24} - \frac{5\pi}{24} + \frac{\pi}{16}\right) + \frac{\pi^3}{48} + \left(- \frac{11}{18} + \frac{\pi}{24}\right) + \frac{16}{27} \Bigg] \Bigg\}$$

$$F^{(5)}(\pi, \beta) = - \frac{N_o}{48} \left\{ \cos \frac{\beta}{2} \left(- \frac{59}{72} - \frac{35\pi}{144} + \frac{\pi^2}{16} - \frac{\pi^3}{96}\right) + \sin \frac{\beta}{2} \right.$$

$$\left. \left(- \frac{49}{216} - \frac{17}{144} + \frac{\pi^2}{16} - \frac{\pi^3}{96}\right) \right\}$$

From equation (26):

$$N_\beta^{(4)} = \frac{N_o}{48} \left[\cos \frac{\beta}{2} \left(\frac{70}{9} + \frac{8}{3} \beta + \frac{\pi\beta}{2} - \frac{\beta^2}{2} - \frac{11\pi}{6}\right) \right.$$

$$\left. + \sin \frac{\beta}{2} \left(- \frac{70}{9} + \frac{8}{3} \beta - \frac{\pi\beta}{2} + \frac{\beta^2}{2} - \frac{5\pi}{6}\right) - \frac{8}{9} \cos \beta \right]$$

Then, from equation (8):

$$N_\beta^{(5)} = 2F^{(5)}(\beta, \beta) - F^{(5)}(0, \beta) - F^{(5)}(\pi, \beta) + N_\beta^{(4)}$$

$$= \frac{N_o}{48} \left\{ - \cos \frac{\beta}{2} \left[\frac{5}{12} - \frac{\pi}{8} + \left(\frac{\pi}{6} - \frac{7}{36}\right) \beta - \left(\frac{1}{24} + \frac{\pi}{16}\right) \beta^2 + \frac{1}{24} \beta^3 \right. \right.$$

$$\left. - \frac{5}{24} + \frac{\pi}{16} + \frac{16}{27} + \frac{59}{72} + \frac{35\pi}{144} - \frac{\pi^2}{16} + \frac{\pi^3}{96} - \left(\frac{70}{9} + \frac{8}{3} \beta + \frac{\pi\beta}{2} - \frac{\beta^2}{2} - \frac{11\pi}{6}\right) \right]$$

$$- \sin\frac{\beta}{2}\left[-\frac{59}{36} - \frac{\pi}{24} + \left(\frac{\pi}{12} - \frac{7}{36}\right)\beta + \left(\frac{1}{24} - \frac{\pi}{16}\right)\beta^2 + \frac{1}{24}\beta^3 - \frac{5}{24} - \frac{\pi}{16}\right.$$

$$\left. + \frac{49}{216} + \frac{17\pi}{144} - \frac{\pi^2}{16} + \frac{\pi^3}{96} - \left(-\frac{70}{9} + \frac{8}{3}\beta - \frac{\pi\beta}{2} + \frac{\beta^2}{2} - \frac{5\pi}{6}\right)\right] + \frac{32}{27}\cos\beta - \frac{8}{9}\cos\beta\Big\} .$$

$$N_\beta^{(5)} = \frac{N_o}{48}\left[\cos\frac{\beta}{2}\left(6\frac{17}{108} - \frac{145\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta}{36} + \frac{\pi\beta}{3} - \frac{11\beta^2}{24} + \frac{\pi\beta^2}{16} - \frac{\beta^3}{24}\right)\right.$$

$$+ \sin\frac{\beta}{2}\left(-6\frac{17}{108} - \frac{61\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta}{36} - \frac{7\pi\beta}{12} + \frac{11\beta^2}{24} + \frac{\pi\beta^2}{16} - \frac{\beta^3}{24}\right)$$

$$\left. + \frac{8}{27}\cos\beta\right]. \tag{28}$$

From equation (9):

$$N_\beta^{(5)} = -\frac{d}{d\beta}N_\beta^{(5)}\Bigg]_{\beta=\alpha} = -\frac{N_o}{48}\left[\cos\frac{\beta}{2}\left(\frac{103}{36} + \frac{\pi}{3} - \frac{22\beta}{24} + \frac{2\pi\beta}{16} - \frac{3\beta^2}{24}\right)\right.$$

$$+ \frac{1}{2}\cos\frac{\beta}{2}\left(-6\frac{17}{108} - \frac{61\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta}{36} - \frac{7\pi\beta}{12} + \frac{11\beta^2}{24} + \frac{\pi\beta^2}{16} - \frac{\beta^3}{24}\right)$$

$$+ \left(\frac{1}{2}\right)\left(-\sin\frac{\beta}{2}\right)\left(6\frac{17}{108} - \frac{145\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta}{36} + \frac{\pi\beta}{3} - \frac{11\beta^2}{24} - \frac{\beta^3}{24} + \frac{\pi\beta^2}{16}\right)$$

$$\left. + \sin\frac{\beta}{2}\left(\frac{103}{36} - \frac{7\pi}{12} + \frac{22\beta}{24} + \frac{2\pi\beta}{16} - \frac{3\beta^2}{24}\right) - \frac{8}{27}\sin\beta\right]_{\beta=\alpha} .$$

$$N_{\alpha}^{(5)} = \frac{N_o}{48}\left[\frac{8}{27}\sin\alpha + \left(\frac{47}{216} + \frac{13\pi}{144} - \frac{\pi^2}{32} + \frac{\pi^3}{192} - \frac{37\alpha}{72} + \frac{\pi\alpha}{6} - \frac{5\alpha^2}{48} - \frac{\pi\alpha^2}{32} + \frac{\alpha^3}{48}\right)\cos\frac{\alpha}{2}\right.$$

$$\left. + \left(\frac{47}{216} - \frac{61\pi}{144} + \frac{\pi^2}{32} - \frac{\pi^3}{192} + \frac{37\alpha}{72} + \frac{\pi\alpha}{24} - \frac{5\alpha^2}{48} - \frac{\alpha^3}{48} + \frac{\pi\alpha^2}{32}\right)\sin\frac{\alpha}{2}\right] \tag{29}$$

D. Summary of Expressions for Distribution over a Semicircular Surface at the n^{th} Collision

1. General Expressions

$$F^{(n)}(\alpha, \beta) \equiv \frac{1}{2}\int N_{\alpha}^{(n-1)}\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)d\alpha. \tag{7}$$

$$N_{\beta}^{(n)} = 2F^{(n)}(\beta, \beta) - F^{(n)}(0, \beta) - F^{(n)}(\pi, \beta) + N_{\beta}^{(n-1)} \tag{8}$$

$$N_{\alpha}^{(n)} = -\frac{d}{d\beta}N_{\beta}^{(n)}\Bigg]_{\beta=\alpha} \tag{9}$$

2. Incident Distribution Uniform over the Surface

$$N_{\alpha}^{(1)} = \frac{N_o}{\pi} \tag{10}$$

$$N_{\beta}^{(1)} = \frac{N_o}{\pi}\left(\frac{\pi}{2} - \beta\right). \tag{11}$$

$$N_{\beta}^{(2)} = \frac{N_o}{\pi}\left(\frac{\pi}{2} - \beta - \cos\frac{\beta}{2} + \sin\frac{\beta}{2}\right) \tag{12}$$

$$N_{\alpha}^{(2)} = \frac{N_o}{2\pi}\left(2 - \sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right). \tag{13}$$

$$N_{\beta}^{(3)} = \frac{N_o}{4\pi}\left[2\pi - 4\beta + \left(\frac{\pi}{2} - \beta + 7\right)\sin\frac{\beta}{2} + \left(\frac{\pi}{2} - \beta - 7\right)\cos\frac{\beta}{2}\right]. \tag{14}$$

$$N_{\alpha}^{(3)} = \frac{N_o}{8\pi}\left[8 - \left(\frac{\pi}{2} - \alpha + 5\right)\cos\frac{\alpha}{2} + \left(\frac{\pi}{2} - \alpha - 5\right)\sin\frac{\alpha}{2}\right]. \tag{15}$$

$$N_{\beta}^{(4)} = \frac{N_o}{16\pi}\left[8\pi - 16\beta + \left(38 + \frac{7\pi}{2} - 8\beta + \frac{\pi\beta}{2} - \frac{\beta^2}{2}\right)\sin\frac{\beta}{2} + \right.$$

$$\left.\left(-38 + \frac{9\pi}{2} - 8\beta - \frac{\pi\beta}{2} + \frac{\beta^2}{2}\right)\cos\frac{\beta}{2}\right]. \tag{16}$$

$$N_{\alpha}^{(4)} = \frac{N_o}{32\pi}\left[32 - \left(22 + \frac{5\pi}{2} - 6\alpha + \frac{\pi\alpha}{2} - \frac{\alpha^2}{2}\right)\cos\frac{\alpha}{2}\right.$$

$$\left.- \left(22 - \frac{7\pi}{2} + 6\alpha + \frac{\pi\alpha}{2} - \frac{\alpha^2}{2}\right)\sin\frac{\alpha}{2}\right]. \tag{17}$$

$$N_{\beta}^{(5)} = \frac{N_o}{64\pi}\left[32\pi - 64\beta + \left(187 + \frac{37\pi}{2} - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta + 5\pi\beta\right.\right.$$

$$\left. - \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6}\right)\sin\frac{\beta}{2} + \left(187 + \frac{57\pi}{2} - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 47\beta - 4\pi\beta\right.$$

$$\left.\left. + \frac{9\beta^2}{2} - \frac{\pi\beta^2}{4} + \frac{\beta^3}{6}\right)\cos\frac{\beta}{2}\right]. \tag{18}$$

$$N_\alpha^{(5)} = \frac{N_o}{128\pi}\left[128 + \left(-93 - \frac{21\pi}{2} + \frac{\pi^2}{4} - \frac{\pi^3}{24} + 29\alpha - 4\pi\alpha + \frac{7\alpha^2}{4} + \frac{\pi\alpha^2}{4} - \frac{\alpha^3}{6}\right)\right.$$

$$\cdot \cos\frac{\alpha}{2} + \left(-93 + \frac{37\pi}{2} - \frac{\pi^2}{4} + \frac{\pi^3}{24} - 29\alpha - 3\pi\alpha + \frac{7\alpha^2}{2}\right.$$

$$\left.\left. - \frac{\pi\alpha^2}{4} + \frac{\alpha^3}{6}\right)\sin\frac{\alpha}{2}\right]. \tag{19}$$

3. Incident Distribution Uniform across the Y-Axis

$$N_\alpha^{(1)} = \frac{1}{2} N_o \sin\alpha. \tag{20}$$

$$N_\beta^{(1)} = \frac{1}{2} N_o \cos\beta. \tag{21}$$

$$N_\beta^{(2)} = \frac{N_o}{3}\left(\cos\frac{\beta}{2} - \sin\frac{\beta}{2} - \frac{1}{2}\cos\beta\right) \tag{22}$$

$$N_\alpha^{(2)} = \frac{N_o}{6}\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} - \sin\alpha\right) \tag{23}$$

$$N_\beta^{(3)} = \frac{N_o}{12}\left[\left(\beta - \frac{\pi}{2} - \frac{5}{3}\right)\sin\frac{\beta}{2} + \left(\beta - \frac{\pi}{2} + \frac{5}{3}\right)\cos\frac{\beta}{2} + \frac{2}{3}\cos\beta\right] \tag{24}$$

$$N_\alpha^{(3)} = \frac{N_o}{12}\left[\left(-\frac{1}{6} - \frac{\pi}{4} + \frac{\alpha}{2}\right)\sin\frac{\alpha}{2} + \left(-\frac{1}{6} + \frac{\pi}{4} - \frac{\alpha}{2}\right)\cos\frac{\alpha}{2} + \frac{2}{3}\sin\alpha\right] \tag{25}$$

$$N_\beta^{(4)} = \frac{N_o}{24}\left[\left(-\frac{35}{9} - \frac{5\pi}{12} + \frac{4\beta}{3} - \frac{\pi\beta}{4} + \frac{\beta^2}{4}\right)\sin\frac{\beta}{2} + \left(\frac{35}{9} - \frac{11\pi}{12} + \frac{4\beta}{3} + \frac{\pi\beta}{4}\right.\right.$$

$$\left.\left. - \frac{\beta^2}{4}\right)\cos\frac{\beta}{2} - \frac{4}{9}\cos\beta\right]. \tag{26}$$

$$N_\alpha^{(4)} = \frac{N_o}{24}\left[\left(\frac{11}{18} - \frac{\pi}{24} - \frac{\alpha}{6} + \frac{\pi\alpha}{8} - \frac{\alpha^2}{8}\right)\cos\frac{\alpha}{2} + \left(\frac{11}{18} - \frac{5\pi}{24} + \frac{\alpha}{6} + \frac{\pi\alpha}{8}\right.\right.$$

$$\left.\left. - \frac{\alpha^2}{8}\right)\sin\frac{\alpha}{2} - \frac{4}{9}\sin\alpha\right]. \tag{27}$$

$$N_\beta^{(5)} = \frac{N_o}{48}\left[\left(\frac{665}{108} - \frac{61\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta}{36} - \frac{7\pi\beta}{12} + \frac{11\beta^2}{24} + \frac{\pi\beta^2}{16} - \frac{\beta^3}{24}\right)\sin\frac{\beta}{2}\right.$$

$$+\left(\frac{665}{108} - \frac{145\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{103\beta}{36} + \frac{\pi\beta}{3} - \frac{11\beta^2}{24} + \frac{\pi\beta^2}{16} - \frac{\beta^3}{24}\right)\cos\frac{\beta}{2}$$

$$\left. + \frac{8}{27}\cos\beta\right]. \tag{28}$$

$$N_\alpha^{(5)} = \frac{N_o}{48}\left[\left(\frac{47}{216} + \frac{13\pi}{144} - \frac{\pi^2}{32} + \frac{\pi^3}{192} - \frac{37\alpha}{72} + \frac{\pi\alpha}{6} - \frac{5\alpha^2}{48} - \frac{\pi\alpha^2}{32} + \frac{\alpha^3}{48}\right)\cos\frac{\alpha}{2}\right.$$

$$+\left(\frac{47}{216} - \frac{61\pi}{144} + \frac{\pi^2}{32} - \frac{\pi^3}{192} + \frac{37\alpha}{72} + \frac{\pi\alpha}{24} - \frac{5\alpha^2}{48} - \frac{\alpha^3}{48} + \frac{\pi\alpha^2}{32}\right)\sin\frac{\alpha}{2}$$

$$\left. + \frac{8}{27}\sin\alpha\right]. \tag{29}$$

SECTION III. DISTRIBUTION OF MOLECULES OVER ELLIPTICAL AND PARABOLIC SURFACES AT FIRST COLLISION - INCIDENT FLUX UNIFORM ACROSS Y-AXIS

For any given geometry, the number of molecules per radian that collide with the surface at an angle α, measured from the negative y-axis, is:

$$N_\alpha \equiv \frac{dN}{d\alpha} = \frac{dN}{dy} \cdot \frac{dy}{d\alpha} .$$

Assuming a uniform incident flux across the y-axis: dN/dy = constant = k, where the constant k is easily determined from the condition that a total of N_o molecules enters the geometry across the opening from y = -1 to y = +1. Then

$$N_o = \int_{-1}^{+1} k\,dy = 2k,$$

or

$$k = \frac{1}{2} N_o,$$

so that:

$$N_\alpha = \frac{1}{2} N_o \frac{dy}{d\alpha} . \tag{30}$$

The derivative $dy/d\alpha$ will depend on the geometry of interest.

A. Ellipse

Consider now a semi-elliptical surface with its opening, two units wide, along the y-axis (Fig. 4). The general equation for such an ellipse is:

$$\frac{x^2}{A^2} + y^2 = 1. \tag{31}$$

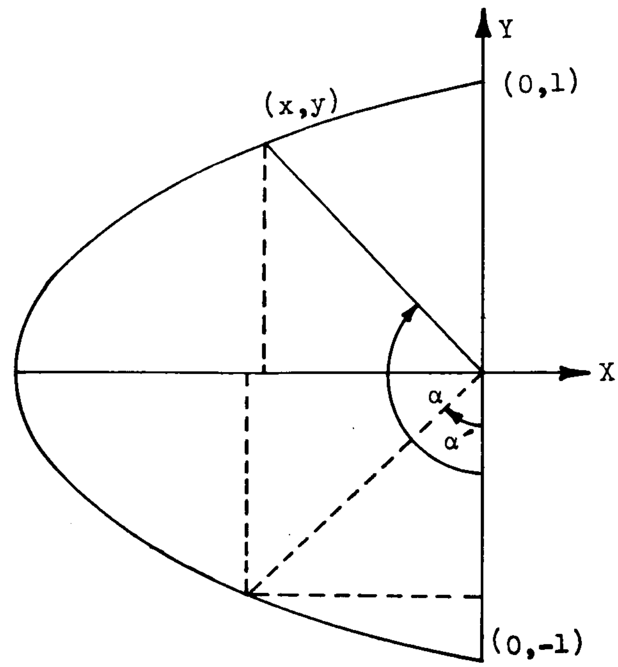

FIGURE 4

(Note that when A = 1, this is the equation for the circle which was treated in the previous section). From Figure 4, for $\alpha > \pi/2$, $\tan(\alpha - \pi/2) = y/-x$, and, since $\tan(\alpha - \pi/2) = -\cot\alpha$, this may be written

$$y = x \cot \alpha.$$

For $\alpha < \pi/2$: $\tan \alpha' = \frac{-x}{-y} = \frac{x}{y}$, so that, in both cases;

$$y = x \cot \alpha \tag{32}$$

and

$$x = y \tan \alpha. \tag{33}$$

Using equation (33) in equation (31):

$$\frac{y^2 \tan^2 \alpha}{A^2} + y^2 = 1,$$

or

$$y^2 \left[\frac{\tan^2 \alpha}{A^2} + 1\right] = 1$$

and

$$y = \frac{\pm A}{(\tan^2 \alpha + A^2)^{1/2}} = \frac{\pm A \cot \alpha}{(1 + A^2 \cot^2 \alpha)^{1/2}} \tag{34}$$

But, when $\cot \alpha \geq 0$, $y \leq 0$ and when $\cot \alpha \leq 0$, $y \geq 0$, so:

$$y = - A \cot \alpha(1 + A^2 \cot^2 \alpha)^{-1/2} .$$

Then

$$\frac{dy}{d\alpha} = (- A \cot \alpha) \left(- \frac{1}{2}\right) \cdot (1 + A^2 \cot^2 \alpha)^{-3/2} (2A^2 \cot \alpha) (- \csc^2 \alpha)$$

$$+ A \csc^2 \alpha (1 + A^2 \cot^2 \alpha)^{-1/2}$$

$$= \csc^2 \alpha \left[\frac{y^3}{\cot \alpha} - \frac{y}{\cot \alpha}\right] = \frac{y \csc^2 \alpha}{\cot \alpha} (y^2 - 1).$$

But, from the equation for the ellipse: $y^2 - 1 = - x^2/A^2$, so that:

$$\frac{dy}{d\alpha} = \frac{y \csc^2 \alpha}{\cot \alpha} \left(\frac{-x^2}{A^2}\right).$$

Also

$$y/\cot \alpha = x,$$

so that

$$\frac{dy}{d\alpha} = \frac{-\,x^3}{A^2 \sin^2 \alpha}$$

But,

$$x^3 = y^3 \tan^3 \alpha,$$

so that

$$\frac{dy}{d\alpha} = \frac{-\,y^3 \tan^3 \alpha}{A^2 \sin^2 \alpha}$$

and

$$y^3 = -\,A^3 \cot^3 \alpha \,(1 + A^2 \cot^2 \alpha)^{-3/2},$$

so that

$$\frac{dy}{d\alpha} = \frac{A^3 \cot^3 \alpha \,(1 + A^2 \cot^2 \alpha)^{-3/2} \tan^3 \alpha}{A^2 \sin^2 \alpha}$$

$$= \frac{A}{\sin^2 \alpha \,(1 + A^2 \cot^2 \alpha)^{3/2}} \,.$$

Multiplying numerator and denominator by $\sin \alpha$ and using equation (30), the final expression becomes:

$$N_\alpha = \frac{1}{2} N_o \, A \sin \alpha \,(\sin^2\alpha + A^2 \cos^2 \alpha)^{-3/2}. \qquad (35)$$

B. Parabola

Consider next a section of a parabola with its opening, two units wide, along the y-axis. See Figure 5. The equation for this parabola is:

$$y^2 = 4Px + 1, \qquad (36)$$

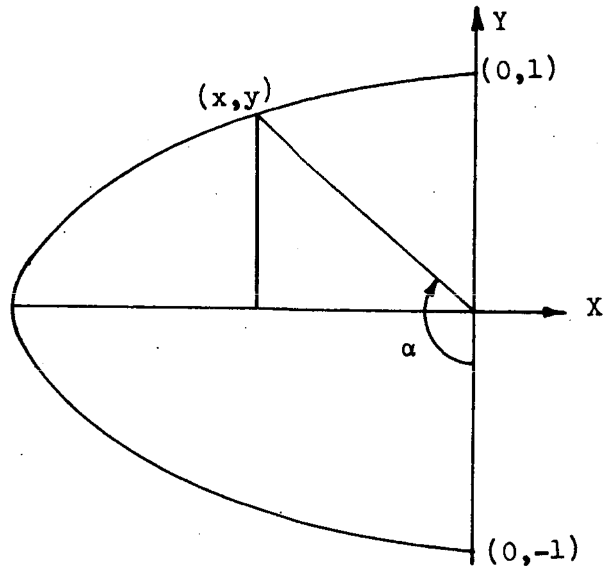

FIGURE 5

with the region of interest being for $x \leq 0$ and $|y| \leq 1$. Note that $y = \pm 1$ when $x = 0$ and $y = 0$ when $x = -\frac{1}{4P}$. As in the case of the ellipse:

$$y = x \cot \alpha. \tag{37}$$

Then, from equation (36):

$$x^2 \cot^2 \alpha - 4Px - 1 = 0,$$

so that

$$x = \frac{4P \pm (16P^2 + 4 \cot^2 \alpha)^{1/2}}{2 \cot^2 \alpha} = \tan^2 \alpha \left[2P \pm (4P^2 + \cot^2 \alpha)^{1/2} \right],$$

but, since $x \leq 0$, $\tan^2 \alpha \geq 0$, and $2P \geq 0$ for all α in the region of interest, the positive square root is discarded, and:

$$x = \tan^2 \alpha \left[2P - (4P^2 + \cot^2 \alpha)^{1/2} \right] \tag{38}$$

so that

$$y = \tan\alpha \left[2P - (4P^2 + \cot^2\alpha)^{1/2}\right].$$

Then

$$\frac{dy}{d\alpha} = \tan\alpha \left[-\frac{1}{2}\ 4P^2 + \cot^2\alpha)^{-1/2}\ (2\cot\alpha)\ (-\csc^2\alpha)\right]$$

$$+ \sec^2\alpha \left[2P - (4P^2 + \cot^2\alpha)^{1/2}\right]$$

$$\frac{dy}{d\alpha} = \frac{1}{\sin^2\alpha\ (4P^2 + \cot^2\alpha)^{1/2}} + \frac{2P - (4P^2 + \cot^2\alpha)^{1/2}}{\cos^2\alpha}$$

$$= \frac{\cos^2\alpha + 2P\sin^2\alpha\ (4P^2 + \cot^2\alpha)^{1/2} - \sin^2\alpha\ (4P^2 + \cot^2\alpha)}{\cos^2\alpha\ \sin^2\alpha\ (4P^2 + \cot^2\alpha)^{1/2}}$$

$$\frac{dy}{d\alpha} = \frac{2P}{\cos^2\alpha}\left[1 - \frac{2P}{(4P^2 + \cot^2\alpha)^{1/2}}\right] \tag{39}$$

$$N_\alpha \equiv \frac{dN}{d\alpha} = \frac{dN}{dy}\frac{dy}{d\alpha} = \frac{N_o P}{\cos^2\alpha}\left[1 - \frac{2P}{(4P^2 + \cot^2\alpha)^{1/2}}\right]. \tag{40}$$

All these expressions are valid throughout the region of interest, except at $\alpha = 0$, π and $\pi/2$. At $\alpha = 0$, π: $\cot\alpha = \infty$, so that x and y are not determined from the equations given above. However, these are just the points at which $x = 0$ and $y = \pm 1$. At $\alpha = \pi/2$ the situation is slightly more complicated, since $N_\alpha \rightarrow 0/0$ as $\alpha \rightarrow \pi/2$. Thus to evaluate N_α at this point, l'Hôpital's Rule may be used as follows. Rewriting equation (39):

$$\frac{dy}{d\alpha} = \frac{2P\left[(4P^2 + \cot^2\alpha)^{1/2} - 2P\right]}{\cos^2\alpha\ (4P^2 + \cot^2\alpha)^{1/2}}.$$

Now let

$$2P\left[(4P^2 + \cot^2 \alpha)^{\frac{1}{2}} - 2P\right] \equiv f(\alpha)$$

$$\cos^2 \alpha \; (4P^2 + \cot^2 \alpha)^{\frac{1}{2}} \equiv g(\alpha)$$

and

$$(4P^2 + \cot^2 \alpha)^{\frac{1}{2}} \equiv Q(\alpha).$$

Then,

$$\frac{dy}{d\alpha} = \frac{f(\alpha)}{g(\alpha)} = \frac{2P\,(Q - 2P)}{Q \cos^2 \alpha}$$

Also

$$Q'(\alpha) = \frac{1}{2}\,(4P^2 + \cot^2 \alpha)^{-\frac{1}{2}}\,(2 \cot \alpha)\,(-\csc^2 \alpha) = \frac{-\cot \alpha}{Q \sin^2 \alpha}$$

and

$$Q''(\alpha) = \frac{Q \sin^2 \alpha\,(\csc^2 \alpha) + \cot \alpha\,(Q \cdot 2 \sin \alpha \cos \alpha + Q' \sin^2 \alpha)}{Q^2 \sin^4 \alpha}$$

$$= \frac{Q + \cos \alpha\,(2Q \cos \alpha + Q' \sin \alpha)}{Q^2 \sin^4 \alpha}$$

so that

$$Q(\pi/2) = 2P$$

$$Q'(\pi/2) = 0$$

$$Q''(\pi/2) = \frac{2P}{4P^2} = \frac{1}{2P}\,.$$

Now

$$\left.\frac{dy}{d\alpha}\right]_{\alpha = \frac{\pi}{2}} = \lim_{\alpha \to \pi/2} \frac{f(\alpha)}{g(\alpha)} = \lim_{\alpha \to \pi/2} \frac{f'(\alpha)}{g'(\alpha)}$$

by l'Hôpital's Rule.

$$f'(\alpha) = 2PQ' \xrightarrow[\alpha \to \pi/2]{} 0$$

since $Q'(\pi/2) = 0$ and $P < \infty$.

$$g'(\alpha) = Q' \cos^2 \alpha + 2 \cos \alpha \, (- \sin \alpha) \, Q \xrightarrow[\alpha \to \pi/2]{} 0$$

since $Q'(\pi/2) = 0$ and $\cos \pi/2 = 0$. Hence,

$$\lim_{\alpha \to \pi/2} \frac{f'(\alpha)}{g'(\alpha)} = \frac{0}{0}$$

and l'Hôpital's Rule may be used again.

$$f''(\alpha) = 2PQ'' \xrightarrow[\alpha \to \pi/2]{} \frac{2P}{2P} = 1$$

$$g''(\alpha) = - 2Q' \cos \alpha \sin \alpha + Q'' \cos^2 \alpha - 2Q \, (\cos^2 \alpha - \sin^2 \alpha) -$$

$$- 2Q' \cos \alpha \sin \alpha \xrightarrow[\alpha \to \pi/2]{} 0 + 0 - 4P(0 - 1) - 0 = 4P$$

so that

$$\left.\frac{dy}{d\alpha}\right]_{\alpha = \frac{\pi}{2}} = \frac{1}{4P}$$

and

$$N_\alpha = \frac{N_0}{8P} \quad \text{for } \alpha = \pi/2. \tag{41}$$

SECTION IV. DISTRIBUTION OF MOLECULES ACROSS THE Y-AXIS UPON EXIT FROM ARRAY

A. Circular Array

1. General Expression

From Section II, page 4, the number of molecules per radian reflected from a point P_α at an angle φ with the normal is:

$$N_{\alpha,\varphi} = \frac{1}{2} N_\alpha \cos \varphi.$$

Now the total number of molecules per radian from P_α that passes through a section of the opening from the origin (0, 0) to point (0, z) is (see Fig. 6):

$$N_{z,\alpha} = \int_0^\delta N_{\alpha,\varphi} \, d\varphi = \int_0^\delta \frac{1}{2} N_\alpha \cos \varphi \, d\varphi = \frac{1}{2} N_\alpha \sin \delta$$

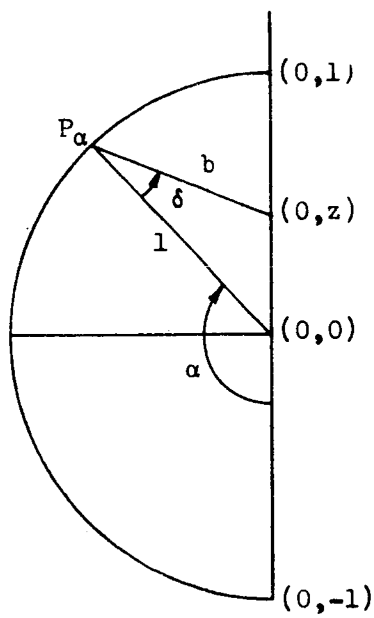

FIGURE 6

and the total number that passes through this section from all points on the surface is:

$$N_z = \int_0^\pi N_{z,\alpha} \; d\alpha = \int_0^\pi \frac{1}{2} N_\alpha \sin\delta \; d\alpha. \tag{42}$$

Thus, to calculate N_z, the initial distribution over the surface, N_α, and $\sin\delta$ must be expressed in terms of α.

In the case of the circle of unit radius, the calculation of $\sin\delta$ is especially simple. From Figure 6:

$$\frac{\sin\delta}{z} = \frac{\sin(\pi - \alpha)}{b}$$

or

$$\sin\delta = \frac{z \sin\alpha}{b}.$$

Also

$$b = \left(1 + z^2 - 2z\cos(\pi - \alpha)\right)^{\frac{1}{2}},$$

and

$$\cos(\pi - \alpha) = -\cos\alpha$$

so that

$$b = (1 + z^2 + 2z\cos\alpha)^{\frac{1}{2}}.$$

Then

$$\sin\delta = \frac{z \sin\alpha}{(1 + z^2 + 2z\cos\alpha)^{\frac{1}{2}}},$$

so that the expression for the distribution, equation (42), becomes:

$$N_z = \frac{z}{2} \int_0^{\pi} \frac{N_\alpha \sin \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}} \quad . \tag{43}$$

2. Incident Distribution Uniform over the Surface

In this case, $N\alpha = \frac{N_o}{\pi}$ = constant, so that:

$$N_z = \frac{N_o z}{2\pi} \int_0^{\pi} \frac{\sin \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}}$$

$$= \frac{N_o z}{2\pi} \left[- \frac{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}}{z} \right]_0^{\pi}$$

$$= - \frac{N_o}{2\pi} \left[(1 + z^2 - 2z)^{\frac{1}{2}} - (1 + z^2 + 2z)^{\frac{1}{2}} \right]$$

$$= - \frac{N_o}{2\pi} \left[\pm (1 - z) - \pm(1 + z) \right]$$

But $(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}} \equiv b$, which is a distance, and therefore positive. Since $0 \leq z \leq 1$, only the positive square roots are meaningful, so that:

$$N_z = \frac{N_o z}{\pi}.$$

Thus the exit distribution is <u>uniform</u> across the y-axis, for molecules exiting the array after the first collision.

3. Incident Distribution Uniform across the Y-Axis

In this case, from equation (21), page :

$$N_\alpha = \frac{1}{2} N_o \sin \alpha \tag{21}$$

so that, from equation (43):

$$N_z = \frac{z}{2} \int_o^\pi \frac{N_\alpha \sin \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}} = \frac{z}{2} \int_o^\pi \frac{\frac{1}{2} N_o \sin^2 \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}}$$

$$N_z = \frac{1}{4} N_o z \int_o^\pi \frac{\sin^2 \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}} . \tag{44}$$

This integral cannot be evaluated in closed form, but has been numerically integrated by a General Electric 225 Computer. The results of this calculation are given in Section VI of this report, for N_o = 10,000 and several values of z.

B. Elliptical Array

Consider now a beam of molecules incident from the right onto a semi-elliptical surface as shown in Figure 7. The calculation of the exit distribution of those molecules which exit the array after one collision will depend upon the relations of the variables y, y_o and z, as follows.

Consider first the case shown in Figure 7, where $y_o \leq 0 \leq z \leq y$. In this case the number of molecules per radian from P_α that passes through a section of the opening from (0, 0) to (0, z) is given by:

$$N_{z,\alpha} = \frac{1}{2} \int_\delta^\epsilon N_\alpha \cos \varphi \, d\varphi = \frac{1}{2} N_\alpha (\sin \epsilon - \sin \delta) \tag{45}$$

where N_α is the incident distribution function defined in Section III, page 43. The total number of molecules that pass through this section from all points on the semi-elliptical surface would then be:

$$N_z = \frac{1}{2} \int_0^\pi N_\alpha (\sin \epsilon - \sin \delta)\, d\alpha, \tag{46}$$

however, care must be taken to consider several different cases, and to define the angles δ and ϵ more precisely. For example, in Figure 8, ϵ is less than δ so that equation (45) would yield a negative result for $N_{z,\alpha}$, the number of molecules passing through the section from (0, 0) to (0, z). This of course must be avoided.

Another ambiguity is introduced when $\sin \delta$ and $\sin \epsilon$ are expressed in terms of α. From Figure 7:

$$\frac{\sin \epsilon}{z - y_o} = \frac{\sin \theta}{b}$$

or

$$\sin \epsilon = \frac{(z - y_o) \sin \theta}{b} .$$

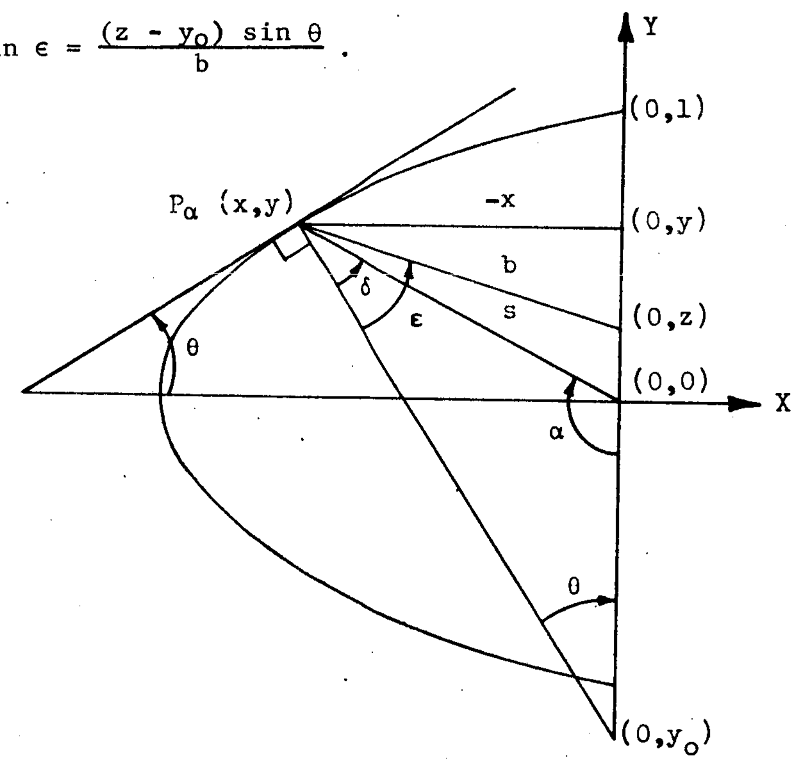

FIGURE 7

But, in the case shown in Figure 9:

$$\frac{\sin \epsilon}{z - y_o} = \frac{\sin (\pi - (-\theta))}{b}$$

or

$$\sin \epsilon = \frac{(z - y_o) \sin (-\theta)}{b}$$

which is the negative of the first case.

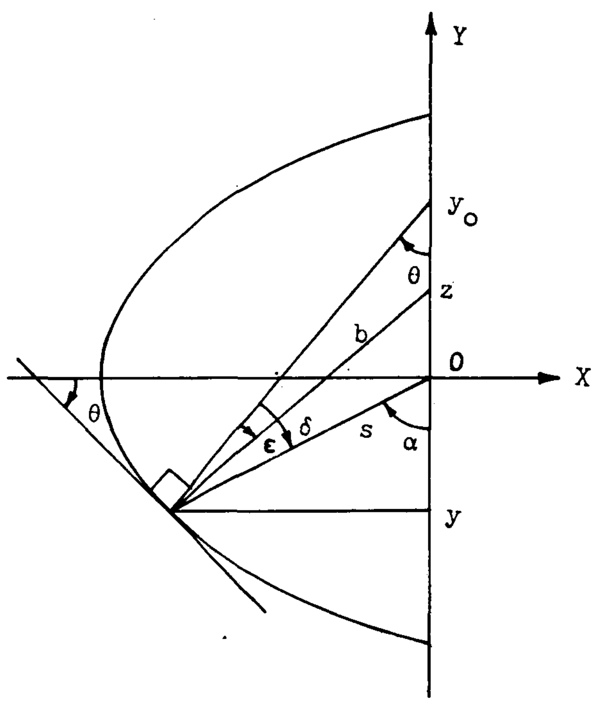

FIGURE 8

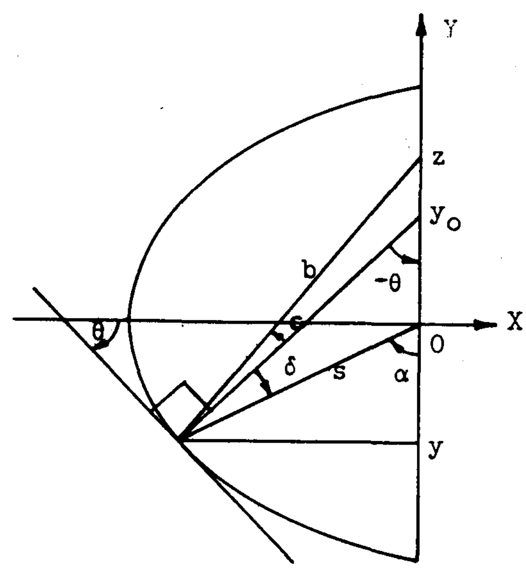

FIGURE 9

Now, a careful examination of Figures 7, 8, and 9 shows that these three cases are the <u>only</u> ones that need be considered, as follows.

In the calculation of sin ϵ and sin δ, only the triangles containing <u>these</u> angles are important; and since the variable y is not a dimension of any of these triangles, its relationship to y_o, z and the origin (0, 0) need not be considered. This reduces the number of

possible cases to <u>six</u>, the number of permutations of <u>three</u> items (y_o, z, and 0). Furthermore, <u>half</u> of the <u>remaining</u> cases are for <u>negative</u> z. Since all the arrays considered in this study are symmetric with respect to the x-axis, all the results for negative z are identical to those for <u>positive</u> z, i.e., $N_{-z} \equiv N_z$. It therefore remains to consider only the three cases shown in Figures 7, 8, and 9.

<u>Case 1.</u>

$$y_o \leq 0 \leq z \qquad \text{(See Figure 7.)}$$

$$N_{z,\alpha} = \frac{1}{2} N_\alpha (\sin \epsilon - \sin \delta) \quad .$$

where

$$\sin \epsilon = \frac{(z - y_o) \sin \theta}{b}$$

and

$$\sin \delta = \frac{- y_o \sin \theta}{s}$$

so that

$$N_{z,\alpha} = \frac{1}{2} N_\alpha \sin \theta \left[\frac{(z - y_o)}{b} + \frac{y_o}{s} \right]. \tag{47}$$

<u>Case 2.</u>

$$0 \leq z \leq y_o \qquad \text{(See Figure 8.)}$$

$$N_{z,\alpha} = \frac{1}{2} N_\alpha (\sin \delta - \sin \epsilon)$$

where now

$$\sin \delta = \frac{y_o \sin (-\theta)}{s} = \frac{- y_o \sin \theta}{s}$$

and

$$\sin \epsilon = \frac{(y_o - z) \sin (-\theta)}{b} = \frac{(z - y_o) \sin \theta}{b}$$

so that

$$N_{z,\alpha} = \frac{1}{2} N_\alpha \sin \theta \left[\frac{-y_o}{s} - \frac{(z - y_o)}{b} \right] \tag{48}$$

which is the negative of the result for Case 1 above.

Case 3.

$$0 \leq y_o \leq z \qquad \text{(See Figure 9.)}$$

$$N_{z,\alpha} = \frac{1}{2} N_\alpha (\sin \delta + \sin \epsilon)$$

where now

$$\sin \delta = \frac{y_o \sin (-\theta)}{s} = \frac{- y_o \sin \theta}{s}$$

and

$$\sin \epsilon = \frac{(z - y_o) \sin (\pi - (-\theta))}{b} = \frac{(y_o - z) \sin \theta}{b}$$

so that

$$N_{z,\alpha} = \frac{1}{2} N_\alpha \sin \theta \left[\frac{-y_o}{s} + \frac{(y_o - z)}{b} \right] \tag{49}$$

which is the same as the result for Case 2.

To take care of the difference in sign between Cases 1, 2, and 3, it is necessary only to insert absolute value signs. Thus, the desired expression for the exit distribution is:

$$N_z = \frac{1}{2} \int_0^{\pi} N_\alpha \left| \left(\frac{y_o - z}{b} - \frac{y_o}{s} \right) \sin \theta \right| d\alpha \tag{50}$$

where the quantities N_α, y_o, b, s and θ must now be expressed in terms of α (z is the independent variable). The incident distribution, N_α, for an ellipse, has been calculated in Section III-A, page 43, for an incident flux uniform across the y-axis. It therefore remains only to calculate y_o, b, s, and θ:

From Figure 7:

$$\tan \theta = - x/(y - y_o)$$

and from Figure 8 or 9:

$$\tan (-\theta) = - x/(y_o - y),$$

so that, in all cases:

$$y_o = y + x \cot \theta. \tag{51}$$

Now the equation for the ellipse is:

$$\frac{x^2}{A^2} + y^2 = 1$$

so that

$$\frac{2x\ dx}{A^2} + 2y\ dy = 0,$$

and

$$\frac{dy}{dx} = \frac{-x}{yA^2};$$

but

$$\frac{dy}{dx} = \tan \theta,$$

so that

$$\tan\theta = -x/yA^2 \tag{52}$$

Also from Figure 7:

$$\tan(\alpha - \tfrac{\pi}{2}) = y/-x,$$

but

$$\tan(\alpha - \tfrac{\pi}{2}) = \frac{\sin(\alpha - \frac{\pi}{2})}{\cos(\alpha - \frac{\pi}{2})} = \frac{-\cos\alpha}{\sin\alpha} = -\cot\alpha,$$

so that

$$\cot\alpha = y/x.$$

From Figure 8 or 9:

$$\tan\alpha = -x/-y,$$

so that in all cases:

and

$$\left.\begin{aligned} x &= y\tan\alpha \\ y &= x\cot\alpha \end{aligned}\right\}. \tag{53}$$

Then from equation (52):

$$\tan\theta = -\tan\alpha/A^2$$

and

$$\theta = \tan^{-1}(-\tan\alpha/A^2) \tag{54}$$

so that, from equations (53) and (51):

$$y_o = y + y \tan\alpha \cot\theta$$

$$= y\left\{1 + \tan\alpha \cdot \cot\left[\tan^{-1}\left(-\tan\alpha/A^2\right)\right]\right\}$$

$$= y\left\{1 + \tan\alpha\left(\frac{A^2}{-\tan\alpha}\right)\right\}$$

$$y_o = y\,(1 - A^2). \tag{55}$$

The expression for y is given in Section III-A, page 45:

$$y = -A \cot\alpha\,(1 + A^2 \cot^2 \alpha)^{-\frac{1}{2}}. \tag{56}$$

Now, from Figure 7:

$$b = +\sqrt{(y - z)^2 + x^2}$$

and from Figure 8 or 9: (57)

$$b = +\sqrt{(z - y)^2 + x^2}$$

Also, from Figure 7, 8, or 9:

$$s = +\sqrt{x^2 + y^2} \tag{58}$$

so that, inserting equations (55), (57) and (58) into equation (50):

$$N_z = \frac{1}{2}\int_0^{\pi} N_\alpha \left|\left(\frac{y(1 - A^2) - z}{\sqrt{(y - z)^2 + x^2}} - \frac{y(1 - A^2)}{\sqrt{x^2 + y^2}}\right)\sin\theta\right| d\alpha \tag{59}$$

where, from page 43:

$$N_\alpha = \frac{1}{2} N_o \; A \sin \alpha \; (\sin^2 \alpha + A^2 \cos^2 \alpha)^{-3/2} \; ;$$

from equation (56):

$$y = - A \cot \alpha \; (1 + A^2 \cot^2 \alpha)^{-\frac{1}{2}} \; ;$$

from equation (53):

$$x = y \tan \alpha;$$

and, from equation (54):

$$\theta = \tan^{-1} (- \tan \alpha / A^2).$$

z is of course the independent variable; A is the constant of the ellipse; N_o is the number of incident molecules.

Now, at $\alpha = 0$ and $\alpha = \pi$, the expression for y (and therefore x) becomes indeterminate, since $\cot 0 = \cot \pi = \infty$. However, from the way in which the ellipses are defined, at $\alpha = 0$, $y = -1$ and $x = 0$, while at $\alpha = \pi$, $y = +1$ and $x = 0$. Also, at $\alpha = \pi/2$, $\tan \alpha = \infty$, but this simply means that $\theta = \pi/2$ also (which can be seen from Figure 7, 8, or 9).

Note that for A = 1, equation (59) reduces to the equation for the circle, as it should:

$$N_\alpha = \frac{1}{2} \sin \alpha \; (\sin^2 \alpha + \cos^2 \alpha)^{-3/2} = \frac{1}{2} N_o \sin \alpha$$

$$y = - \cot \alpha \; (1 + \cot^2 \alpha)^{-\frac{1}{2}}$$

$$= - \cot \alpha \left(1 + \frac{\cos^2 \alpha}{\sin^2 \alpha}\right)^{-\frac{1}{2}}$$

$$= - \cot \alpha \left(\frac{\sin^2 \alpha + \cos^2 \alpha}{\sin^2 \alpha} \right)^{-\frac{1}{2}}$$

$$= - \cot \alpha \, (\sin \alpha)^{-1}$$

$$y = - \cos \alpha$$

$$x = y \tan \alpha = - \cos \alpha \tan \alpha$$

$$x = - \sin \alpha$$

$$\theta = \tan^{-1} (- \tan \alpha)$$

$$\tan \theta = - \tan \alpha$$

$$\sin \theta = \tan \theta \sqrt{\frac{1}{1 + \tan^2 \theta}} = - \tan \alpha \, (1 + \tan^2 \alpha)^{-\frac{1}{2}}$$

$$= - \tan \alpha \left(1 + \frac{\sin^2 \alpha}{\cos^2 \alpha} \right)^{-\frac{1}{2}} = - \tan \alpha \, (\pm \cos \alpha)$$

$$\sin \theta = \pm \sin \alpha,$$

but since only the absolute value enters into the calculation:

$$\sin \theta = \sin \alpha$$

so that:

$$N_z = \frac{1}{2} \int_0^{\pi} \frac{1}{2} N_o \sin \alpha \left| \frac{- z \sin \alpha}{\sqrt{(- \cos \alpha - z)^2 + \sin^2 \alpha}} \right| d\alpha$$

$$= \frac{1}{4} N_o z \int_0^{\pi} \frac{\sin^2 \alpha \, d\alpha}{\sqrt{\cos^2 \alpha + 2z \cos \alpha + z^2 + \sin^2 \alpha}}$$

$$= \frac{1}{4} N_o z \int_0^{\pi} \frac{\sin^2 \alpha \, d\alpha}{(1 + z^2 + 2z \cos \alpha)^{\frac{1}{2}}}$$

which is identical to equation (44), page 54.

Equation (59) for the distribution of molecules exiting a semi-elliptical array after one collision is not integrable in closed form, but has been numerically integrated by the GE 225 Computer, for an incident flux uniform across the y-axis. The results are given in Section VI of this report for N_o = 10,000 and several values of A and z.

C. Parabolic Array

By referring to Figure 7 of the previous section, it can be seen that the expression for the exit distribution from a parabolic section will be very similar to that for the semi-elliptical array, except that the expressions for x, y, θ, y_o and N_α will be different, as follows.

The equation for the parabola is:

$$y^2 = 4Px + 1,$$

and from Section III-B, page 46:

$$y = x \cot \alpha \tag{60}$$

so that

$$x^2 \cot^2 \alpha - 4Px - 1 = 0$$

and

$$x = \frac{4P \pm (16P^2 + 4\cot^2\alpha)^{\frac{1}{2}}}{2\cot^2\alpha}$$

$$x = \frac{2P - (4P^2 + \cot^2\alpha)^{\frac{1}{2}}}{\cot^2\alpha} \qquad (61)$$

the negative sign being chosen because the domain of interest is for $x \leq 0$ and $P \geq 0$. Again using the equation for the parabola:

$$y^2 = 4Px + 1$$

$$2ydy = 4Pdx$$

$$\tan\theta = \frac{dy}{dx} = \frac{2P}{y}$$

$$\theta = \tan^{-1}(2P/y). \qquad (62)$$

Now,

$$y_o \doteq y + x\cot\theta \qquad (63)$$

as before, and the expression for N_α (for incident beam uniform across the y-axis) is given by equation (40) page 48. The expression for the exit distribution is then given by

$$N_z = \frac{1}{2}\int_o^\pi \frac{N_o P}{\cos^2\alpha}\left[1 - \frac{2P}{(4P^2 + \cot^2\alpha)^{\frac{1}{2}}}\right]\left[\frac{y_o - z}{\sqrt{(y-z)^2 + x^2}} - \frac{y_o}{\sqrt{x^2 + y^2}}\right]|\sin\theta|d\alpha$$

(64)

where x, y, y_o and θ are given by equations (61), (60), (63) and (62) above. z, of course, is the independent variable; P is the constant of the parabola; N_o is the number of incident molecules. As in the case of the ellipse, care must be taken at $\alpha = 0, \pi/2, \pi$. At $\alpha = 0, \pi$: $\cot \alpha = \infty$, but again $x = 0$ and $y = \pm 1$ at these points. At $\alpha = \pi/2$, it has already been shown in Section III-B, page 46, that $N_\alpha = N_o/8P$.

Again the expression for N_z is not integrable in closed form, but results of the numerical integration are given in Section VI of this report.

SECTION V. DISTRIBUTION OF MOLECULES ALONG THE CENTER LINE (X-AXIS) OF AN ELLIPSE AND A PARABOLA

A. General Expressions

The distribution of molecules along the center line of an ellipse (or parabola or hyperbola) can be found by calculating the number of molecules which passes through a section of the x-axis between x_1 and x_2 (Fig. 10).

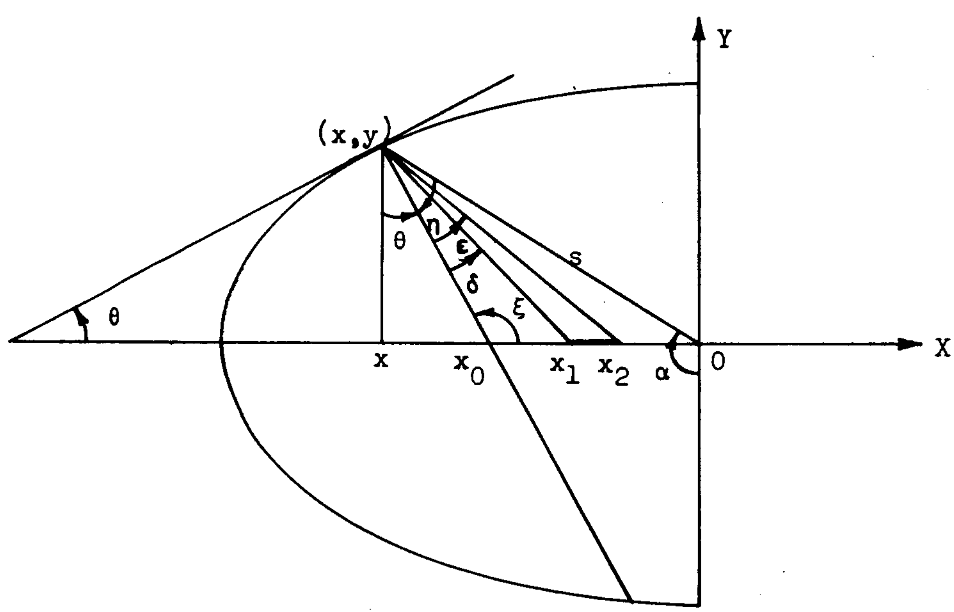

FIGURE 10

Using the same argument as in Section IV (page 51), this number would be given by:

$$N_x = \int_{\pi/2}^{\pi} N_\alpha (\sin \epsilon - \sin \delta)\, d\alpha \tag{65}$$

if now the angles ϵ and δ are defined in Figure 10. Since all the arrays considered are symmetric with respect to the x-axis, the integration extends from $\pi/2$ to π and is doubled to obtain the total contribution.

The problem now is to express $\sin \epsilon$ and $\sin \delta$ in terms of the independent variables α and x_1. For purposes of numerical evaluation of the integral, x_2 has been taken to be $x_1 + A/8$ for the ellipse and $x_1 + 1/48P$ for the parabola., The initial distribution function, N_α, has been calculated for the ellipse and parabola in Section III above.

Now, from Figure 10, it is clear that there will again be a number of cases to consider, depending upon the relative positions of x, x_o, x_1 and x_2. (The position of the origin does not matter, since none of the pertinent triangles contain the origin.) If all permutations of these variables were to be considered, there would be $4! = 24$ possible cases. Fortunately, this number can be reduced, as follows: (a) as mentioned above, x^2 is always greater than x_1. This eliminates half of the 24 cases. (b) An examination of ellipses and parabolas shows that $x < x_o$ in all cases. Since this condition is independent of (a), 6 of the remaining 12 cases are eliminated. There are, therefore, six cases to consider:

1. $x \leq x_1 \leq x_2 \leq x_o$
2. $x \leq x_1 \leq x_o \leq x_2$
3. $x \leq x_o \leq x_1 \leq x_2$
4. $x_1 \leq x_2 \geq x \leq x_o$
5. $x_1 \leq x \leq x_2 \leq x_o$
6. $x_1 \leq x \leq x_o \leq x_2$.

It will now be convenient to introduce the following definitions:

$$
\begin{aligned}
a &\equiv x_o - x_1 \\
b &\equiv + [y^2 + (x_o - x)^2]^{\frac{1}{2}} \\
c &\equiv x_o - x_2 \\
d &\equiv + [y^2 + (x_1 - x)^2]^{\frac{1}{2}} \\
r &\equiv + [y^2 + (x_2 - x)^2]^{\frac{1}{2}}
\end{aligned}
\qquad (66)
$$

<u>Case 1: $x \leq x_1 \leq x_2 \leq x_o$ (Fig. 11)</u>

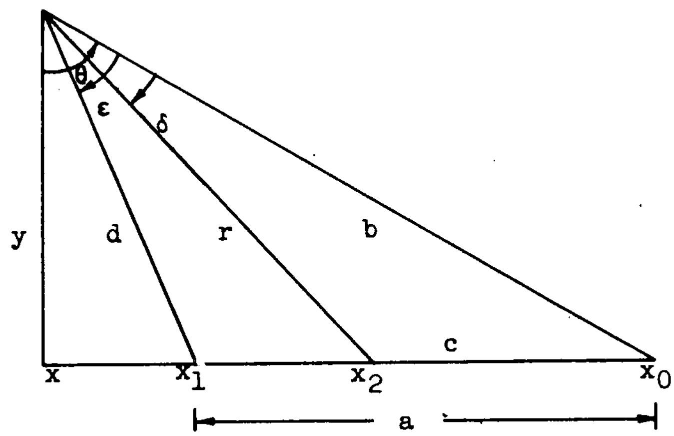

FIGURE 11

From Figure 11:

$$\frac{\sin \epsilon}{a} = \frac{\sin (\pi/2 - \theta)}{d}$$

or

$$\sin \epsilon = \frac{a \cos \theta}{d} .$$

Also

$$\frac{\sin \delta}{c} = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{r} .$$

The desired quantity is

$$(\sin \epsilon - \sin \delta) = \left(\frac{a}{d} - \frac{c}{r}\right) \cos \theta .$$

Expressions must now be obtained for a, c, d, r and θ in terms of the independent variables α, x_1 and x_2. From the definitions of a, c, d and r (see above), this means that x_o, x and y (and θ) must be expressed in terms of α, x_1 and x_2. But this has already been done for x, y and θ in Section IV for the ellipses and parabolas. The remaining quantity, x_o, will be calculated later (see page 72).

<u>Case 2. $x \le x_1 \le x_o \le x_2$ (See Figure 12).</u>

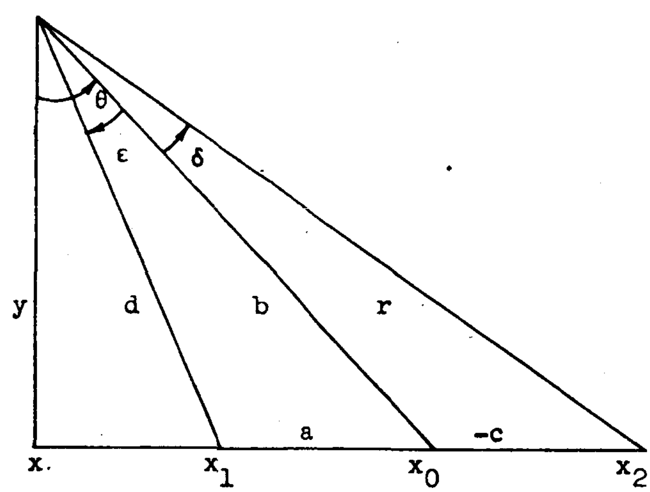

FIGURE 12

From Figure 12:

$$\frac{\sin \epsilon}{a} = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{d}$$

or

$$\sin \epsilon = \frac{a \cos \theta}{d} .$$

Also

$$\frac{\sin \delta}{- c} = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{r}$$

or

$$\sin \delta = - \frac{c \cos \theta}{r} .$$

Now, from Figure 12, it is clear that the desired quantity in this case is <u>not</u> ($\sin \epsilon - \sin \delta$), but is ($\sin \epsilon + \sin \delta$). This may be obtained by using the same expression as in Case 1 above for ($\sin \epsilon - \sin \delta$), namely:

$$\left(\frac{a}{d} - \frac{c}{r}\right) \cos \theta,$$

the negative sign being taken care of, since c is negative in this case. Note that all angles are considered positive, regardless of their direction.

<u>Case 3. $x \leq x_0 \leq x_1 \leq x_2$ (See Figure 13).</u>

$$\frac{\sin \epsilon}{a} = \frac{\sin\left(\frac{\pi}{2} + \theta\right)}{d}$$

or

$$\sin \epsilon = - \frac{a \cos \theta}{d}$$

$$\frac{\sin \delta}{-c} = \frac{\sin\left(\frac{\pi}{2} + \theta\right)}{r}$$

or

$$\sin \delta = -\frac{c \cos \theta}{r}$$

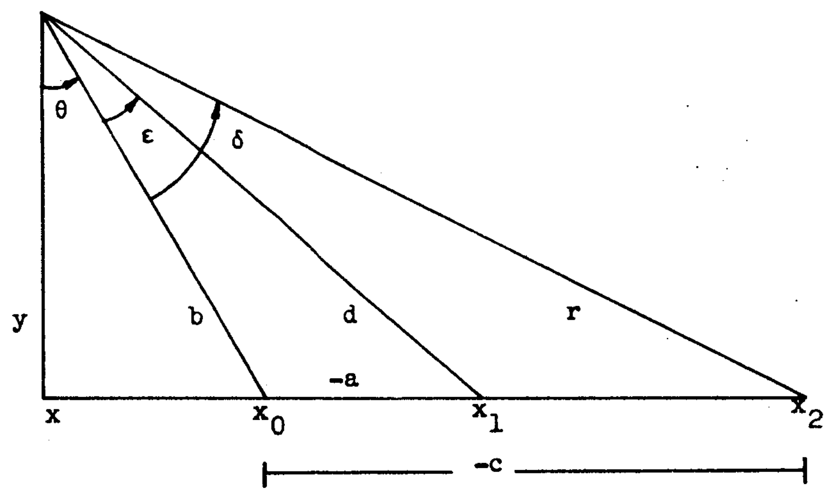

FIGURE 13

In this case, the desired quantity is $(\sin \delta - \sin \epsilon)$. Again, if the original (Case 1) expression is used, the correct answer is obtained, i.e.,

$$\left(\frac{a}{d} - \frac{c}{r}\right) \cos \theta = (\sin \delta - \sin \epsilon)$$

<u>Case 4. $x_1 \leq x_2 \leq x \leq x_0$.</u>

At this point it is clear that the relative position of x does not enter into the calculation of $(\sin \epsilon - \sin \delta)$. Case 4 is

therefore the same as Case 1. Similarly, Cases 5 and 6 are the same as Cases 1 and 2, respectively. Thus, the desired expression is, for all cases:

$$(\sin \epsilon - \sin \delta) = \left(\frac{a}{d} - \frac{c}{r}\right) \cos \theta. \tag{67}$$

The only remaining task is to calculate x_o in terms of α; referring back to Figure 10:

$$\frac{\sin \eta}{-x_o} = \frac{\sin \xi}{s}$$

or

$$x_o = -\frac{s \sin \eta}{\sin \xi},$$

but

$$\theta + \eta + \frac{\pi}{2} + \left(\alpha - \frac{\pi}{2}\right) = \pi$$

so that

$$\eta = \pi - (\alpha + \theta).$$

Then

$$\sin \eta = \sin (\alpha + \theta).$$

Also

$$\sin \xi = \sin \left(\frac{\pi}{2} + \theta\right) = \cos \theta$$

and

$$s = (x^2 + y^2)^{1/2},$$

so that

$$x_o = - \frac{(x^2 + y^2)^{1/2} \sin (\alpha + \theta)}{\cos \theta} \tag{68}$$

Equation (65), with the appropriate substitutions from equations (66), (67) and (68) and the expressions for N_α, θ, x and y from Section III, yields the distribution N_x along the x-axis as a function of x_1. Calculations for specific cases follow below.

B. Ellipse

From page 44, equations (32), (33) and (34):

$$x = - A (1 + A^2 \cot^2 \alpha)^{-1/2} ,$$

and

$$y = x \cot \alpha.$$

From page 60, equation (54):

$$\theta = \tan^{-1} (- \tan \alpha/A^2).$$

From page 46, equation (35):

$$N_\alpha = \frac{1}{2} N_o A \sin \alpha(\sin^2 \alpha + A^2 \cos^2 \alpha)^{-3/2}$$

where N_o is an arbitrary number of incident molecules and A is the "shape parameter" of the ellipse:

$$\frac{x^2}{A^2} + y^2 = 1.$$

Summarizing, then: (x_1, x_2, A, α, N_o are independent variables).

$$\theta = \tan^{-1}\left(-\frac{\tan\alpha}{A^2}\right)$$

$$x = -A\,(1 + A^2 \cot^2 \alpha)^{-1/2}$$

$$y = x \cot \alpha$$

$$N_\alpha = \frac{1}{2} N_o\, A \sin \alpha(\sin^2 \alpha + A^2 \cos^2 \alpha)^{-3/2}$$

$$x_o = -\,(x^2 + y^2)^{1/2} \sin\,(\alpha + \theta)/\cos\,\theta$$

$$a = x_o - x_1$$

$$b = [y^2 + (x_o - x)^2]^{1/2}$$

$$c = x_o - x_2$$

$$d = [y^2 + (x_1 - x)^2]^{1/2}$$

$$r = [y^2 + (x_2 - x)^2]^{1/2}$$

$$N_x = \int_{\pi/2}^{\pi} N_\alpha \left[\left(\frac{a}{d} - \frac{c}{r}\right) \cos\,\theta\right] d\alpha.$$

C. Parabola

From page 47 , equations (38) and (37):

$$x = \tan^2 \alpha \, [2P - (4P^2 + \cot^2 \alpha)^{1/2}]$$

$$y = x \cot \alpha.$$

From page 48, equation (40):

$$N_\alpha = \frac{N_o P}{\cos^2 \alpha} \, [1 - 2P \, (4P^2 + \cot^2 \alpha)^{-1/2}]$$

where P is the shape parameter of the parabola

$$y^2 = 4Px + 1.$$

Summarizing:

$$x = \tan^2 \alpha \, [2P - (4P^2 + \cot^2 \alpha)^{1/2}]$$

$$y = x \cot \alpha$$

$$\theta = \tan^{-1} \, (2P/y)$$

$$N_\alpha = \frac{N_o P}{\cos^2 \alpha} \, [1 - 2P \, (4P^2 + \cot^2 \alpha)^{-1/2}].$$

The remaining steps are identical to those for the ellipse.

SECTION VI. NUMERICAL RESULTS

A. Fraction of Molecules Making n^{th} Collision with a Semi-Circular Surface.

From the expressions derived for distributions over the semi-circular surface (page 39), the total number of molecules making the n^{th} collision (with the surface) can be found by integrating $N_{\alpha}^{(n)}$ from $\alpha = 0$ to π. It is easier, however, to calculate $2 \times N_{\beta}^{(n)}$ at $\beta = 0$, which is the same quantity (see Figure 1, page 2). The results follow:

1. Incident Distribution Uniform over the Surface

$$2 \times N_{\beta=0}^{(1)} = 2 \times \frac{N_o}{\pi} (\pi/2) = N_o$$

$$2 \times N_{\beta=0}^{(2)} = 2 \times \frac{N_o}{\pi} \left(\frac{\pi}{2} - 1\right) = N_o \left(\frac{\pi - 2}{\pi}\right) \cong 0.3634\ N_o$$

$$2 \times N_{\beta=0}^{(3)} = 2 \times \frac{N_o}{4\pi} \left[2\pi + \left(-7 + \frac{\pi}{2}\right)\right] = N_o \left(\frac{5\pi - 14}{4\pi}\right) \cong 0.1359\ N_o$$

$$2 \times N_{\beta=0}^{(4)} = 2 \times \frac{N_o}{16\pi} \left[8\pi + \left(-38 + \frac{9\pi}{2}\right)\right] = N_o \left(\frac{25\pi - 76}{16\pi}\right) \cong 0.0505\ N_o$$

$$2 \times N_{\beta=0}^{(5)} = 2 \times \frac{N_o}{64\pi} \left[32\pi + \left(-187 + \frac{57\pi}{2} - \frac{\pi^2}{4} + \frac{\pi^3}{24}\right)\right] \cong 0.0188\ N_o.$$

2. Incident Distribution Uniform across the Y-Axis

$$2 \times N_{\beta=0}^{(1)} = 2 \times \frac{N_o}{2} = N_o$$

$$2 \times N_{\beta=0}^{(2)} = 2 \times \frac{N_o}{3} \left(1 - \frac{1}{2}\right) = \frac{N_o}{3} \cong 0.3333\ N_o$$

$$2 \times N_{\beta=0}^{(3)} = 2 \times \frac{N_o}{12}\left(\frac{5}{3} - \frac{\pi}{2} + \frac{2}{3}\right) = N_o\left(\frac{14 - 3\pi}{36}\right) \cong 0.1271\ N_o$$

$$2 \times N_{\beta=0}^{(4)} = 2 \times \frac{N_o}{48}\left(7\frac{7}{9} - \frac{11\pi}{6} - \frac{8}{9}\right) = \frac{N_o}{24}\left(6\frac{8}{9} - \frac{11\pi}{6}\right) \cong 0.0471\ N_o$$

$$2 \times N_{\beta=0}^{(5)} = 2 \times \frac{N_o}{48}\left(6\frac{17}{108} - \frac{145\pi}{72} + \frac{\pi^2}{16} - \frac{\pi^3}{96} + \frac{8}{27}\right) \cong 0.0175\ N_o$$

From these results, the <u>fraction</u> of molecules exiting the array after n collisions can be calculated:

$$\alpha_n = \frac{(\#\ \text{making}\ n^{th}\ \text{collision}) - (\#\ \text{making}\ (n+1)^{th}\ \text{collision})}{(\#\ \text{making}\ n^{th}\ \text{collision})}.$$

The results of the trivial calculation are given in Table I.

B. <u>Distributions across Openings and Center-Lines of Arrays after One Collision, with Incident Distribution Uniform Across Y-Axis.</u>

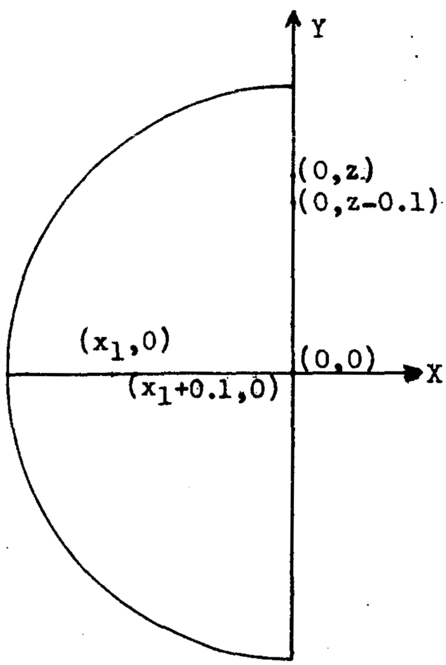

FIGURE 14

In the analysis of the preceding sections, expressions were derived for the following:

N_z = number of molecules that pass through a section of the opening between (0, 0) and (0, z), after one collision with the surface.

N_x = number of molecules that pass through a section of the x-axis between (x_1, 0) and (x_1 + 0.1, 0), after one collision with the surface.

Programs were written for a General Electric 225 Computer to evaluate the integrals for these expressions, for several different geometries. The results of these calculations are given in Tables II through V, and are shown graphically in Figures 15 through 30.

In Tables II and III (for exit distributions), the independent variable z is given in column (A), and N_z is given in column (B). But the quantity of greater interest is the number between z and z - 0.1. This number, $N_{\triangle z}(225)$, is given in column (C). The numbers in column (D), $N_{\triangle z}(7090)$, are the corresponding values obtained by the Monte Carlo method, using the IBM 7090 Computer[1]. The difference between these two, $\triangle$, is given in column (E) and the percentage difference in column (G). The probable error, P. E., given in column (F), is calculated as follows.

In general, if N measurements of a quantity are taken and p is the probability of obtaining a given measurement, then the standard deviation of a measurement is given by:

$$\sigma = \sqrt{N_p(1 - p)},$$

and the probable error, P. E., is given by P. E. = 0.6745σ. The probability, p, is found from the 225 results (see next paragraph) and N = 10,000 is the number of "measurements." For example, the first value of $N_{\triangle z}$ (for z = 0.1) in Table II is 957. This means that the probability, p, of a molecule falling into the section between z = 0 and z = 0.1 is 957/10,000 = 0.0957 (since the incident number of molecules is 10,000). Then (1 - p) = 0.9042 and

$$\sigma = \sqrt{(10,000)\ (0.0957)\ (0.9042)} = \sqrt{865.3194} = 29.42,$$

[1]This method and the results have been reported in Marshall Technical Paper MTP-AERO-62-53 by J. O. Ballance, W. K. Roberts, and D. W. Tarbell, and were presented at the Cryogenic Engineering Conference, August 14-16, 1962.

so that

$$P. E. = (0.6745)\ (29.42) = 19.8 \cong 20.$$

In Tables IV and V (for center-line distributions), the independent variable, x, is given in column (A) and $N_x(225)$ in column (B). Since N_x is already for a fixed interval, no subtraction is necessary. $N_x(7090)$ is given for the same interval in column (C); the difference, $\triangle$, in column (D); the % difference in (F), and the probable error, P. E., in column (E).

In all cases, the GE 225 Program employed the Simpson Rule to evaluate the integral. Since the accuracy of these evaluations depends on the size of the increment, $\triangle\alpha$, used in the Simpson Rule, this increment was taken to be just small enough so that the GE 225 result contained as many significant figures as the 7090 result. The value used for $\triangle\alpha$ ranged from 1° down to 0.005°, depending on the particular case; in each case, the increment used was made smaller until two results were identical (to the same number of significant figures as there were in the 7090 result). The numbers given in column (C) for $N_{\triangle z}(225)$ can therefore be considered exact, with the error in the Monte Carlo results only.

SECTION VII. DISCUSSIONS AND CONCLUSIONS

A. Surface Distributions

The distribution of molecules on the inner surface of a semicircle (or a long cylinder in three dimensions) has been derived as a function of the number of collisions made by the molecules. Initial distributions considered were for completely uniform incidence upon the surface and for a directed, uniform flow on the entrance to the surface. Figure 31 graphically presents the percentage of particles colliding per radian for a semicircle and the two initial distributions considered. It is seen that, for either initial distribution, after two collisions the distribution on the surface becomes approximately constant. Of course, the percentage of particles making the third, fourth, and fifth collisions becomes quite small (i.e., 1.75% make 5 collisions). Chahine [5] has derived the total number flux incident at any point on a semicircular surface per unit time and unit area with a directed uniform flow. By summing the distributions for the five collisions this same flux may be found. This summation is within 2% of the Chahine result. Since approximately 99.4% of the incident particles have exited after 5 collisions, this difference is not significant and shows that, for this case, less than 2% of the flux results from particles making more than 5 collisions.

B. Exit and Center Line Distributions

The distributions of particles exiting the surface array and along the center line of symmetry of the array have been derived. Due to the complexity of these derivations, analytical solutions were obtained only for single collisions with the surface; however, computer programs can consider any number of collisions. Early in the study it was thought that concentration of particles (focusing) would occur at the exit of the array. No such focusing was observed, but the distributions at the exit and along the center line does yield very interesting results. For example, for an elliptical surface with the parameter A equal to two, it is seen from the Table II and Figure 18 that 50% of the incident particles cross the inner 50% of the center line after the first collision. A practical application of the result is shown in Figure 22. Here is a cryogenic pumping array which could be used for highly directed flow fields such as in a low density wind tunnel, a rocket exhaust test chamber, or a rocket sounding probe for atmospheric sampling. Radiation shields reduce the radiant heat losses of the 20°K surface and also provides some impedance to molecules from leaving the array. Collection efficiency (that is, the percentage of particles condensed by the 20°K surface compared to the total number of particles entering the array) should be very high (hopefully, 80-90 percent).

C. Monte Carlo Computer Techniques

The applicability and accuracy of Monte Carlo computer techniques for the study of free molecular flow has been shown. These techniques are necessary for this type of study since the analytical solutions become quite complex. As future plans in the study are for extension of the program into transition flow where intermolecular collisions will be considered, it is felt that only through the use of Monte Carlo methods can meaningful studies be made.

APPENDIX A

Useful Integrals

$$\int \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\,d\alpha = \sin^2\frac{\alpha}{2} \tag{1}$$

$$\int \sin^2\frac{\alpha}{2}\,d\alpha = \frac{1}{2}(\alpha - \sin\alpha) \tag{2}$$

$$\int \cos^2\frac{\alpha}{2}\,d\alpha = \frac{1}{2}(\alpha + \sin\alpha) \tag{3}$$

$$\int \alpha\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\,d\alpha = \frac{1}{2}(\sin\alpha - \alpha\cos\alpha) \tag{4}$$

$$\int \alpha^2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\,d\alpha = \alpha\sin\alpha + \cos\alpha - \frac{1}{2}\alpha^2\cos\alpha \tag{5}$$

$$\int \sin\alpha\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\,d\alpha = \frac{1}{4}(\alpha - \frac{1}{2}\sin 2\alpha) \tag{6}$$

$$\int \cos\alpha\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\,d\alpha = \frac{1}{4}\sin^2\alpha \tag{7}$$

$$\int \alpha\cos^2\frac{\alpha}{2}\,d\alpha = \frac{1}{2}(\frac{1}{2}\alpha^2 + \alpha\sin\alpha + \cos\alpha) \tag{8}$$

APPENDIX A (Cont'd)

$$\int \alpha^2 \cos^2 \frac{\alpha}{2}\, d\alpha = \frac{\alpha^3}{6} + \left(\frac{\alpha^2}{2} - 1\right) \sin \alpha + \alpha \cos \alpha \qquad (9)$$

$$\int \sin \alpha \cos^2 \frac{\alpha}{2}\, d\alpha = - \cos^4 \frac{\alpha}{2} \qquad (10)$$

$$\int \cos \alpha \cos^2 \frac{\alpha}{2}\, d\alpha = \frac{1}{4}\left(\alpha + \frac{3}{2} \sin \alpha + \sin \alpha \cos^2 \frac{\alpha}{2} + \frac{1}{4} \sin 2\alpha\right) \qquad (11)$$

$$\int \alpha \sin^2 \frac{\alpha}{2}\, d\alpha = \frac{1}{2}\left(\frac{\alpha^2}{2} - \alpha \sin \alpha - \cos \alpha\right) \qquad (12)$$

$$\int \alpha^2 \sin^2 \frac{\alpha}{2}\, d\alpha = \frac{\alpha^3}{6} - \left(\frac{\alpha^2}{2} - 1\right) \sin \alpha - \alpha \cos \alpha \qquad (13)$$

$$\int \sin\alpha \sin^2 \frac{\alpha}{2}\, d\alpha = \sin^4 \frac{\alpha}{2} \qquad (14)$$

$$\int \cos \alpha \sin^2 \frac{\alpha}{2}\, d\alpha = \frac{1}{4}\left(- \alpha + \frac{3}{2} \sin \alpha + \sin \alpha \cos^2 \frac{\alpha}{2} - \frac{3}{4} \sin 2\alpha\right) \qquad (15)$$

$$\int \cos \frac{\alpha}{2}\, d\alpha = 2 \sin \frac{\alpha}{2} \qquad (16)$$

$$\int \sin \frac{\alpha}{2}\, d\alpha = - 2 \cos \frac{\alpha}{2} \qquad (17)$$

APPENDIX A (Cont'd)

$$\int \alpha \sin \frac{\alpha}{2} d\alpha = 4 \sin \frac{\alpha}{2} - 2\alpha \cos \frac{\alpha}{2} \tag{18}$$

$$\int \alpha \cos \frac{\alpha}{2} d\alpha = 4 \cos \frac{\alpha}{2} + 2\alpha \sin \frac{\alpha}{2} \tag{19}$$

$$\int \frac{\alpha^2}{2} \cos \frac{\alpha}{2} d\alpha = 4\alpha \cos \frac{\alpha}{2} + \left(\alpha^2 - 8\right) \sin \frac{\alpha}{2} \tag{20}$$

$$\int \frac{\alpha^2}{2} \sin \frac{\alpha}{2} d\alpha = 4\alpha \sin \frac{\alpha}{2} - \left(\alpha^2 - 8\right) \cos \frac{\alpha}{2} \tag{21}$$

$$\int \alpha^3 \cos \frac{\alpha}{2} d\alpha = (12\alpha^2 - 96) \cos \frac{\alpha}{2} + (2\alpha^3 - 48\alpha) \sin \frac{\alpha}{2} \tag{22}$$

$$\int \alpha^3 \sin \frac{\alpha}{2} d\alpha = (12\alpha^2 - 96) \sin \frac{\alpha}{2} - (2\alpha^3 - 48\alpha) \cos \frac{\alpha}{2} \tag{23}$$

TABLE I

FRACTION OF MOLECULES THAT EXIT SEMICIRCULAR ARRAY AFTER THE n^{th} COLLISION

n	A number making n^{th} collision $\div N_o$	B number exiting array after n^{th} collision $\div N_o$	$\alpha_n = \frac{B}{A}$
1	1.0000	0.6366	0.6366
2	0.3634	0.2275	0.6260
3	0.1359	0.0854	0.6284
4	0.0505	0.0317	0.6277
5	0.0188	-	-
1	1.0000	0.6667	0.6667
2	0.3333	0.2062	0.6187
3	0.1271	0.0800	0.6294
4	0.0471	0.0296	0.6285
5	0.0175	-	-

TABLE II

ELLIPSE EXIT DISTRIBUTION

(A)	(B)	(C)	(D)	(E)	(F)	(G)
Z	N_z	$N_{\Delta z}(225)$	$N_{\Delta z}(7090)$	Δ	P.E.	Δ(%)
A = 0.50						
0.1	956.5	957	936	21	20	2.2
0.2	1909.7	953	924	29	20	3.1
0.3	2856.1	946	961	15	20	1.6
0.4	3791.3	935	934	1	20	0.1
0.5	4710.1	919	924	5	19	0.5
0.6	5604.9	895	893	2	19	0.2
0.7	6464.7	860	935	25	19	2.7
0.8	7271.1	806	827	21	18	2.5
0.9	7989.5	718	691	27	17	3.9
1.0	8527.5	538	529	9	15	1.7
A = 1.00						
0.1	784.4	784	805	21	18	2.7
0.2	1562.9	779	729	50	18	6.4
0.3	2329.4	767	791	24	18	3.1
0.4	3077.4	748	797	49	18	6.6
0.5	3800.1	723	744	21	17	2.9
0.6	4489.5	689	640	49	17	7.1
0.7	5136.3	647	624	23	17	3.6
0.8	5728.8	593	568	25	16	4.2
0.9	6249.9	521	477	44	15	8.4
1.0	6666.7	417	433	16	13	3.8

TABLE II (Cont'd)

(A)	(B)	(C)	(D)	(E)	(F)	(G)
Z	N_z	$N_{\triangle z}(225)$	$N_{\triangle z}(7090)$	$\triangle$	P.E.	$\triangle(\%)$
A = 1.50						
0.1	600.7	601	597	4	16	0.7
0.2	1196.7	596	583	13	16	2.2
0.3	1783.1	586	583	3	16	0.5
0.4	2355.2	572	676	104	16	15.4
0.5	2907.8	553	554	1	15	0.2
0.6	3435.8	528	557	29	15	5.2
0.7	3933.2	497	500	3	15	0.6
0.8	4393.2	460	483	23	14	4.8
0.9	4806.7	414	421	7	13	1.7
1.0	5157.6	351	348	3	12	0.9
A = 2.00						
0.1	460.5	461	486	25	14	5.4
0.2	917.8	457	468	11	14	2.4
0.3	1368.6	451	452	1	14	0.2
0.4	1809.7	441	418	23	14	5.2
0.5	2237.7	428	415	13	14	3.0
0.6	2649.1	411	385	26	13	6.3
0.7	3040.1	391	375	16	13	4.1
0.8	3406.3	366	361	5	12	1.4
0.9	3742.1	336	334	2	12	0.6
1.0	4037.7	296	288	8	11	2.7

TABLE III

PARABOLA EXIT DISTRIBUTION

(A)	(B)	(C)	(D)	(E)	(F)	(G)
z	N_z	$N_{\Delta z}(225)$	$N_{\Delta z}(7090)$	Δ	P.E.	Δ(%)
P = 0.5000						
0.1	942.1	942	968	26	20	2.8
0.2	1879.6	938	930	8	20	0.9
0.3	2808.1	929	941	12	20	1.3
0.4	3723.5	915	889	26	19	2.8
0.5	4621.5	898	927	29	19	3.2
0.6	5498.7	877	866	11	19	1.3
0.7	6351.4	853	816	37	19	4.3
0.8	7176.4	825	848	23	19	2.8
0.9	7970.2	794	766	28	18	3.5
1.0	8727.6	757	771	14	18	1.8
P = 0.2500						
0.1	748.5	749	732	17	18	2.3
0.2	1492.1	744	721	23	18	3.1
0.3	2225.9	734	734	0	18	0
0.4	2945.4	720	715	5	17	0.7
0.5	3646.6	701	701	0	17	0
0.6	4325.7	679	678	1	17	0.1
0.7	4979.4	654	645	9	17	1.4
0.8	5604.7	625	608	17	16	2.7
0.9	6198.7	594	611	17	16	2.9
1.0	6756.8	558	547	11	15	2.0

TABLE III (Cont'd)

(A)	(B)	(C)	(D)	(E)	(F)	(G)
Z	N_z	$N_{\Delta z}(225)$	$N_{\Delta z}(7090)$	Δ	P.E.	$\Delta(\%)$
P = 0.1250						
0.1	439.4	439	453	14	14	3.2
0.2	877.0	438	422	16	14	3.7
0.3	1310.9	434	403	31	14	7.1
0.4	1739.5	429	428	1	14	0.2
0.5	2161.1	422	394	28	14	6.6
0.6	2574.0	413	392	21	13	5.1
0.7	2976.7	403	415	12	13	3.0
0.8	3367.6	391	393	2	13	0.5
0.9	3745.0	377	420	43	13	11.4
1.0	4106.6	362	400	38	12	10.5
P = 0.0833						
0.1	288.3	288	283	5	11	1.7
0.2	575.9	288	279	9	11	3.1
0.3	862.0	286	275	11	11	3.8
0.4	1146.0	284	264	20	11	7.0
0.5	1427.0	281	265	16	11	5.7
0.6	1704.5	278	258	20	11	7.2
0.7	1977.6	273	260	13	11	4.8
0.8	2245.6	268	291	23	11	8.6
0.9	2507.6	262	256	6	11	2.3
1.0	2762.3	255	272	17	11	6.7

TABLE IV

ELLIPSE CENTER LINE DISTRIBUTION

(A)	(B)	(C)	(D)	(E)	(F)
X	$N_x(225)$	$N_x(7090)$	Δ	P.E.	$\Delta(\%)$
A = 0.5000					
-0.5000	405	403	2	13	0.5
-0.4375	385	373	12	13	3.1
-0.3750	364	393	29	13	8.0
-0.3125	343	358	15	12	4.4
-0.2500	322	301	21	12	6.5
-0.1875	301	266	35	12	11.6
-0.1250	280	265	15	11	5.4
-0.0625	260	278	18	11	6.9
A = 1.0000					
-1.000	872	866	21	19	2.5
-0.875	785	793	8	18	1.0
-0.750	698	677	21	17	2.4
-0.625	616	640	24	16	2.6
-0.500	538	566	28	15	2.8
-0.375	465	438	27	14	3.0
-0.250	399	389	10	13	3.3
-0.125	340	336	4	12	3.5

TABLE IV (Cont'd)

(A)	(B)	(C)	(D)	(E)	(F)
X	$N_x(225)$	$N_x(7090)$	Δ	P.E.	$\Delta(\%)$
A = 1.5000					
-1.5000	1318	1299	65	22	1.8
-1.3125	1109	1082	27	21	1.9
-1.1250	928	959	31	20	2.2
-0.9375	771	738	33	18	2.3
-0.7500	634	618	16	16	2.5
-0.5625	515	541	26	15	2.9
-0.3750	412	394	18	13	3.2
-0.1875	326	333	7	12	3.7
A = 2.0000					
-2.00	1719	1695	17	25	1.5
1.75	1345	1370	25	23	1.7
1.50	1067	1002	65	21	2.0
1.25	847	866	19	19	2.2
1.00	668	676	8	17	2.5
0.75	518	525	7	15	2.9
0.50	393	372	21	13	3.3
0.25	293	314	21	11	3.8

TABLE V

PARABOLA CENTER LINE DISTRIBUTION

(A) X	(B) $N_x(225)$	(C) $N_x(7090)$	(D) $\triangle$	(E) P.E.	(F) $\triangle(\%)$
P = 0.5000					
-0.50000	284	260	24	11	8.5
-0.45833	276	243	33	11	12.0
-0.41666	267	298	31	11	11.6
-0.37500	257	250	7	11	2.7
-0.33333	248	261	13	10	5.2
-0.29166	238	219	19	10	8.0
-0.25000	229	268	39	10	7.0
-0.20833	220	214	6	10	2.7
-0.16666	210	212	2	10	1.0
-0.12500	201	198	3	9	1.5
-0.08333	193	179	14	9	7.3
-0.04166	184	188	4	9	2.2
P = 0.250					
-1.0000	615	589	26	16	4.2
-0.9166	568	571	3	16	0.5
-0.8333	525	549	24	15	4.6
-0.7500	487	489	2	15	0.4
-0.6666	453	477	24	14	5.3
-0.5833	421	408	13	14	3.1
-0.5000	391	409	18	13	4.6
-0.4166	362	334	28	13	7.7
-0.3333	335	346	11	12	3.3
-0.2500	308	303	5	12	1.6
-0.1666	282	261	21	11	7.4
-0.0833	258	238	20	11	7.8

TABLE V (Cont'd)

(A) X	(B) $N_x(225)$	(C) $N_x(7090)$	(D) Δ	(E) P.E.	(F) $\Delta(\%)$
P = 0.1250					
-2.000	1169	1116	53	22	4.5
-1.833	941	924	17	20	1.8
-1.666	798	812	14	18	1.8
-1.500	699	724	25	17	3.6
-1.333	623	642	19	16	3.0
-1.166	562	615	53	16	9.4
-1.000	509	475	34	15	6.7
-0.833	461	458	3	14	0.7
-0.666	414	445	31	13	7.5
-0.500	368	371	3	13	0.8
-0.333	322	326	4	12	1.2
-0.166	276	250	26	11	9.4
P = 0.0833					
-3.0000	1564	1553	11	25	2.0
-2.75	1101	1131	30	21	2.7
-2.50	886	880	6	19	0.7
-2.25	756	780	24	18	3.2
-2.00	666	654	12	17	1.8
-1.75	598	614	16	16	2.7
-1.50	541	520	21	15	3.9
-1.25	491	463	28	15	5.7
-1.00	443	454	11	14	2.5
-0.75	392	423	31	13	7.9
-0.50	336	324	12	12	3.6
-0.25	274	285	11	11	4.0

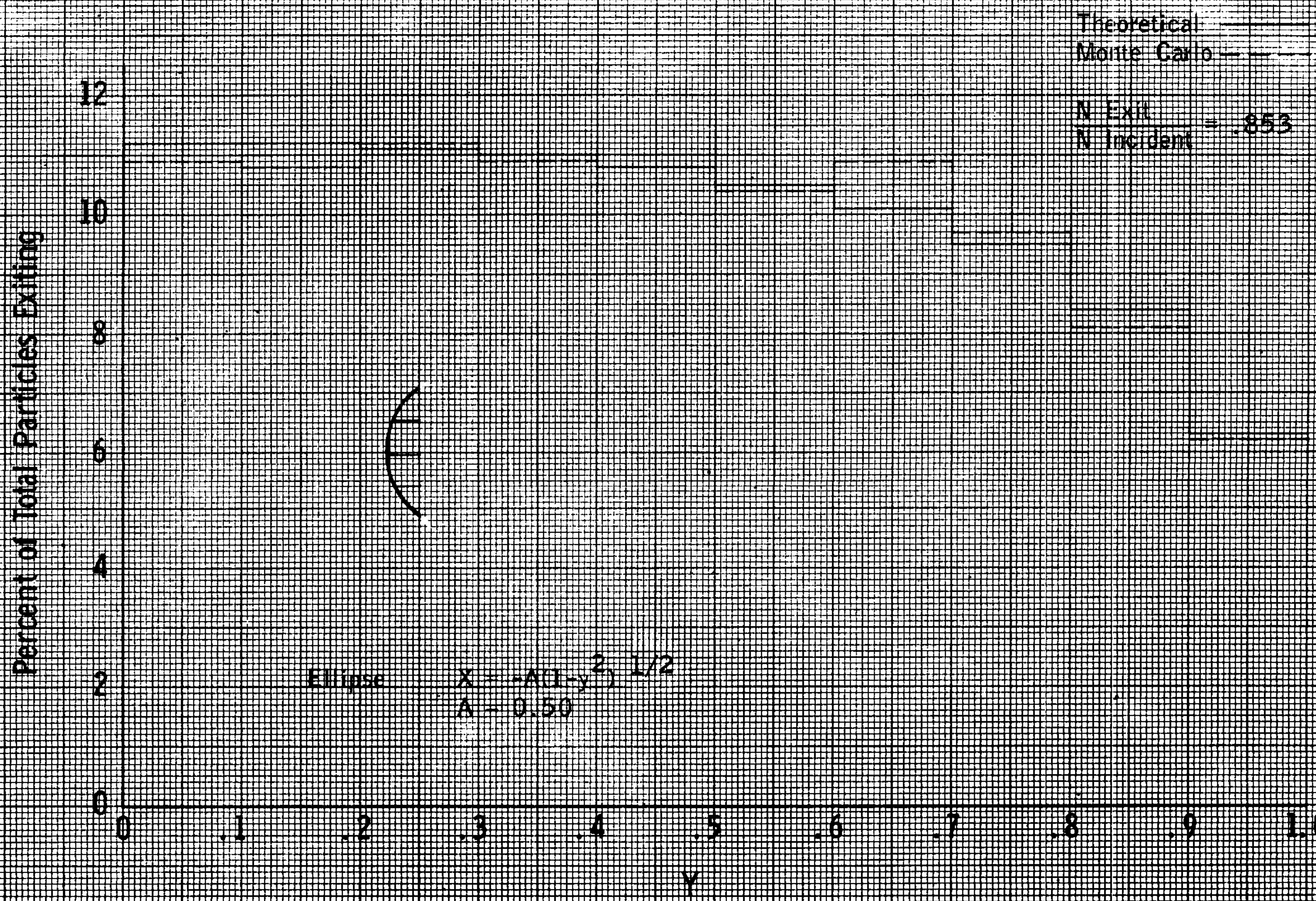

Figure 15 Distribution of Particles Exiting After Single Collision (Incident Distribution Uniform Across Opening)

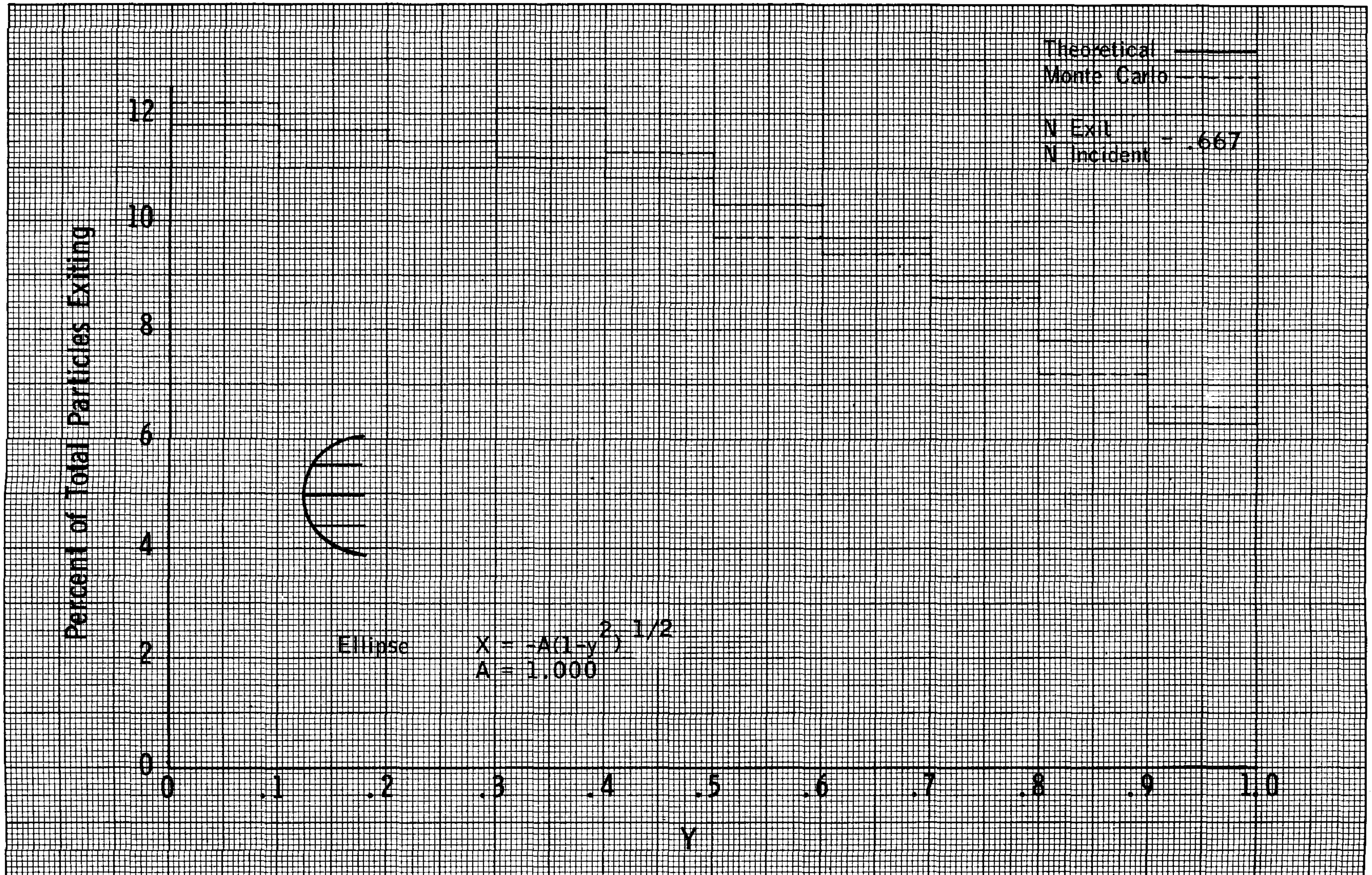

Figure 16 Distribution of Particles Exiting After Single Collision (Incident Distribution Uniform Across Opening)

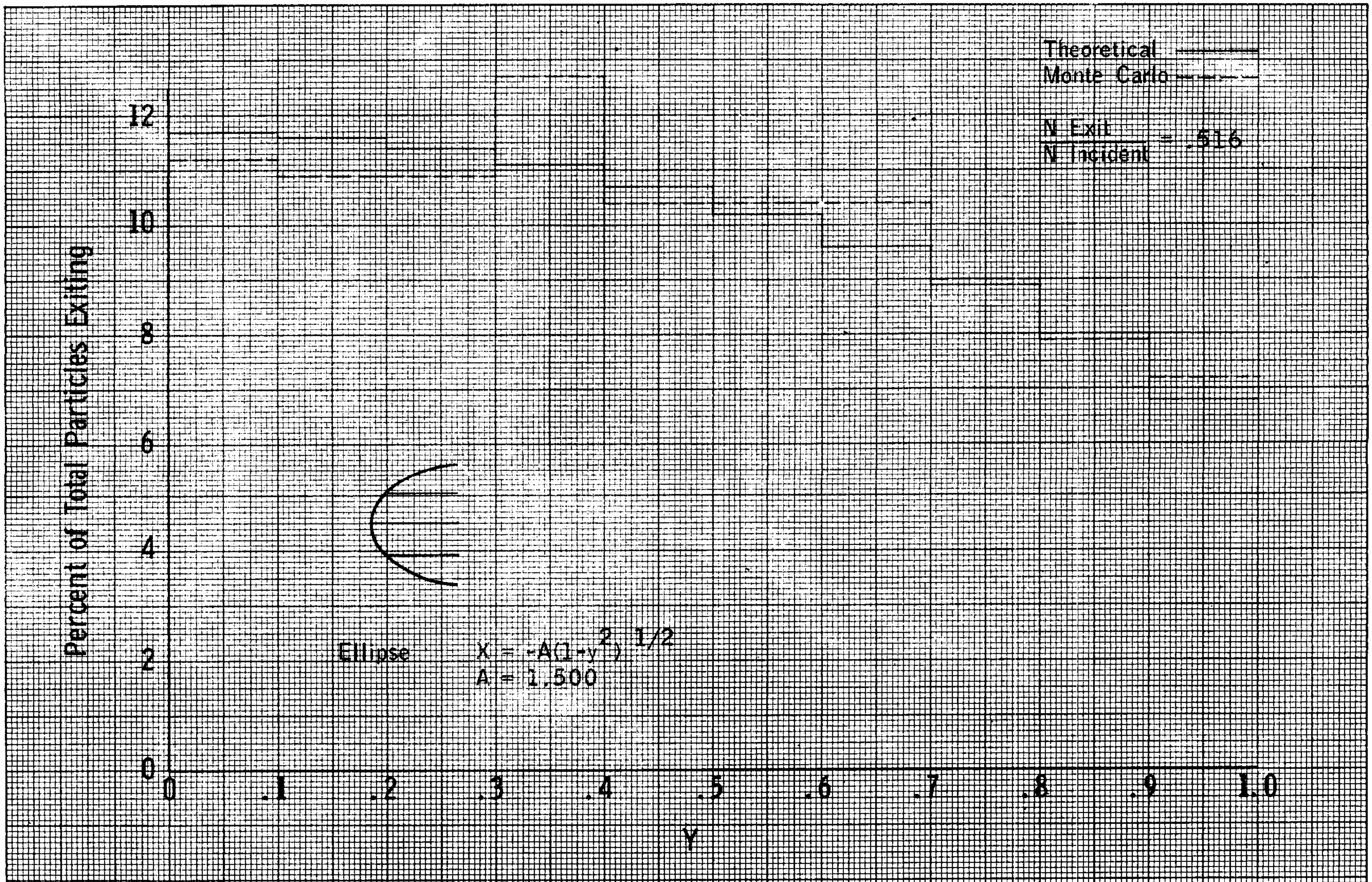

Figure 17 Distribution of Particles Exiting After Single Collision
(Incident Distribution Uniform Across Opening)

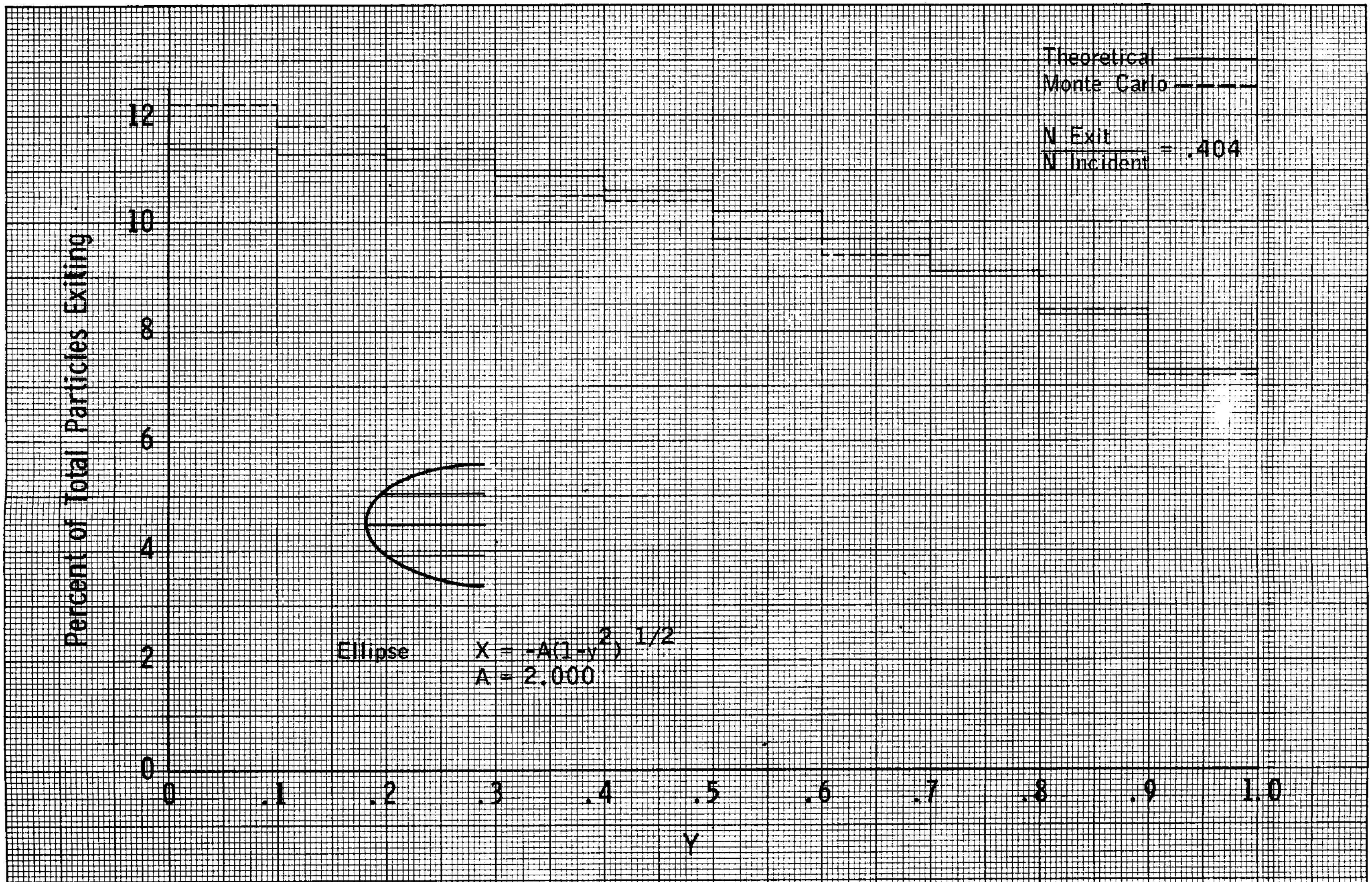

Figure 18 Distribution of Particles Exiting After Single Collision (Incident Distribution Uniform Across Opening)

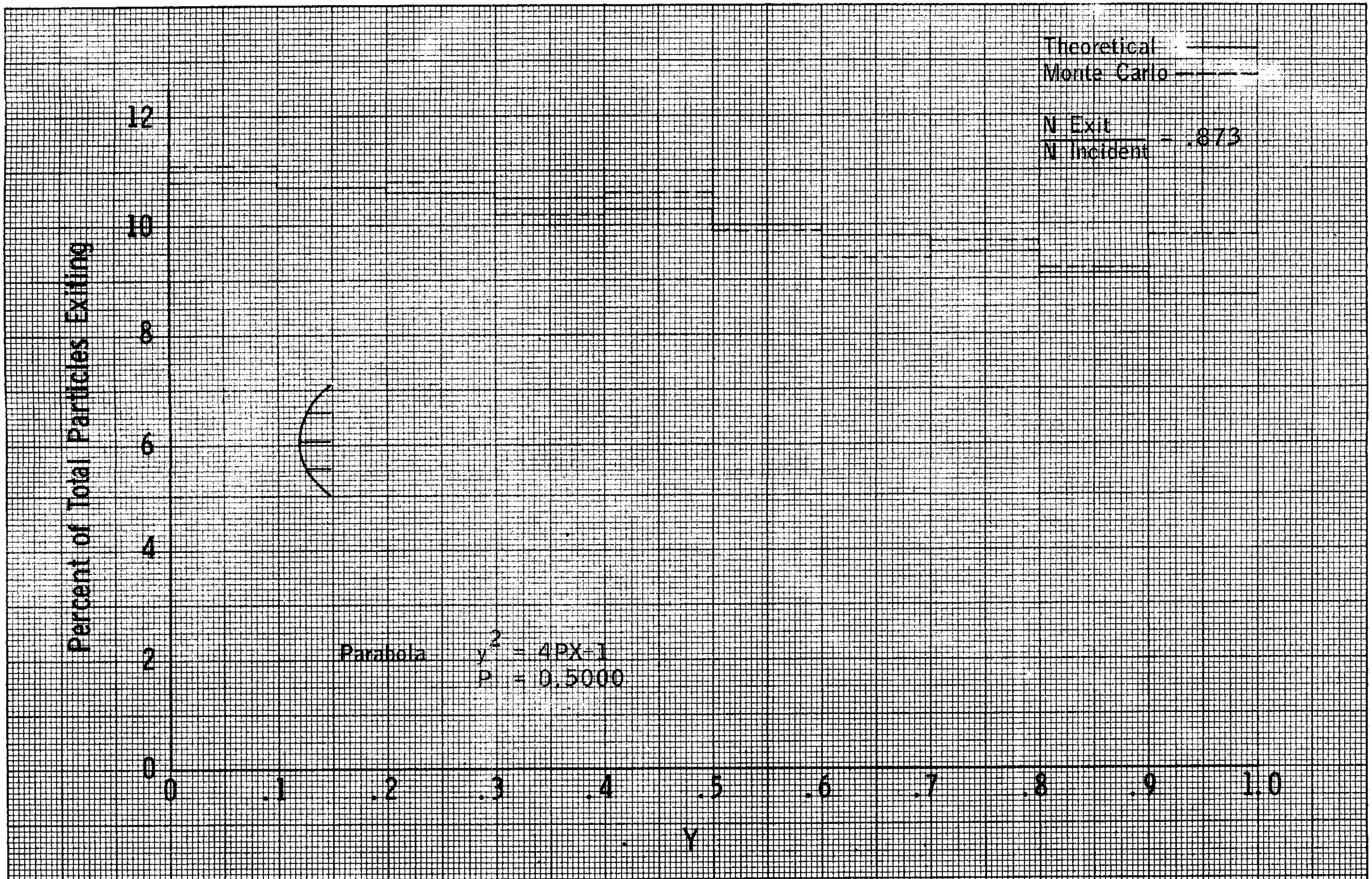

Figure 19 Distribution of Particles Exiting After Single Collision (Incident Distribution Uniform Across Opening)

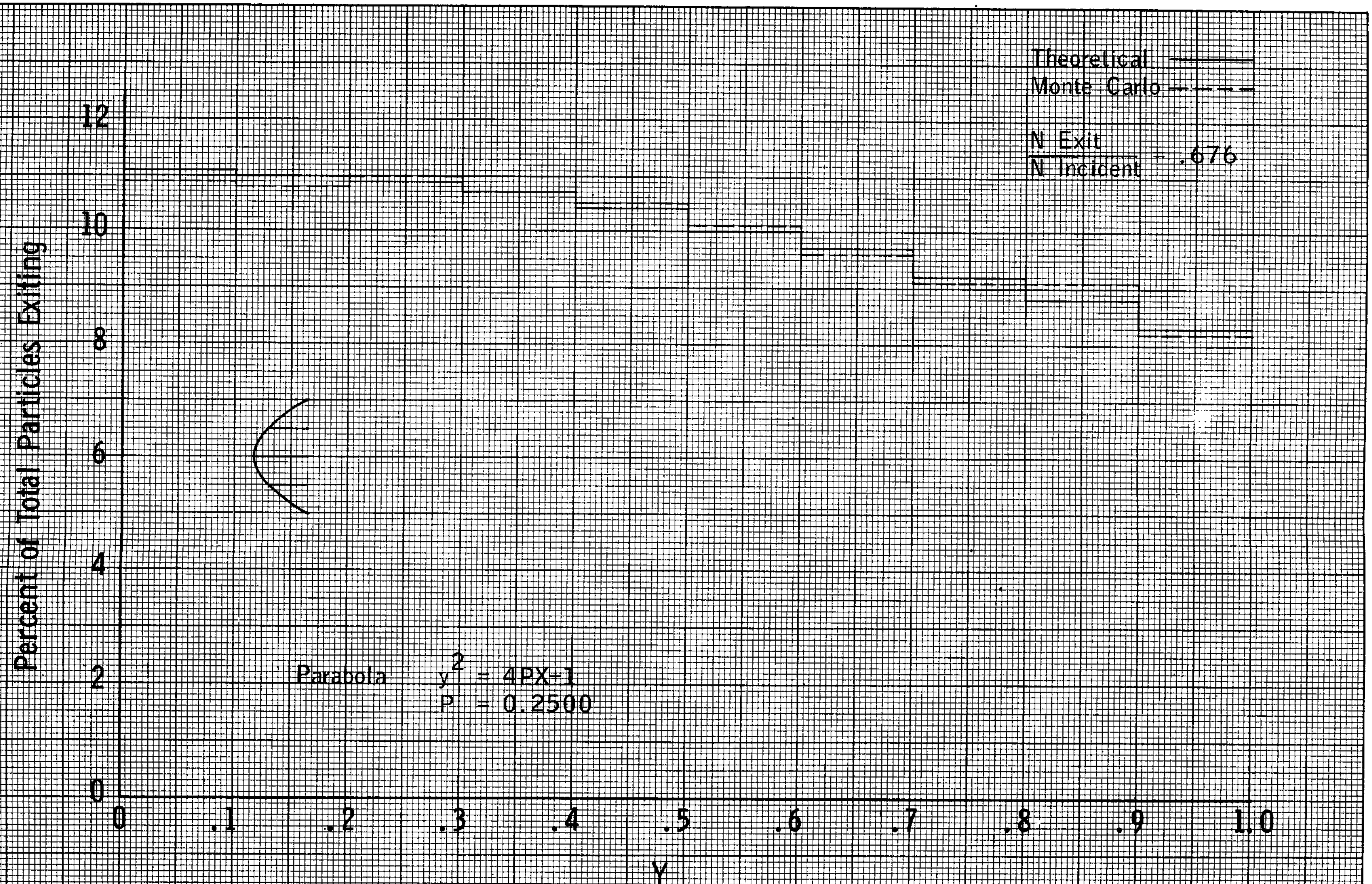

Figure 20 Distribution of Particles Exiting After Single Collision (Incident Distribution Uniform Across Opening)

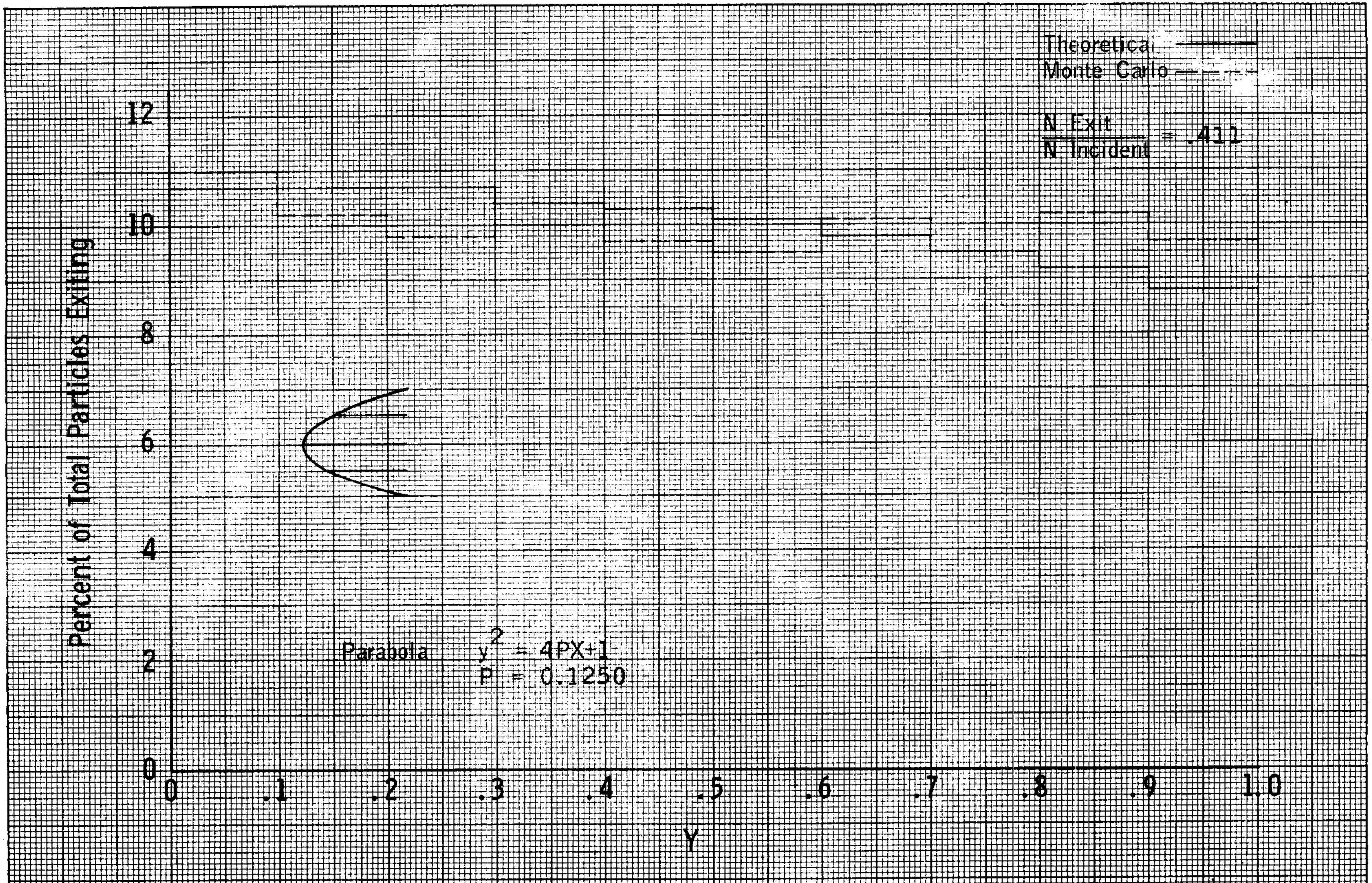

Figure 21 Distribution of Particles Exiting After Single Collision (Incident Distribution Uniform Across Opening)

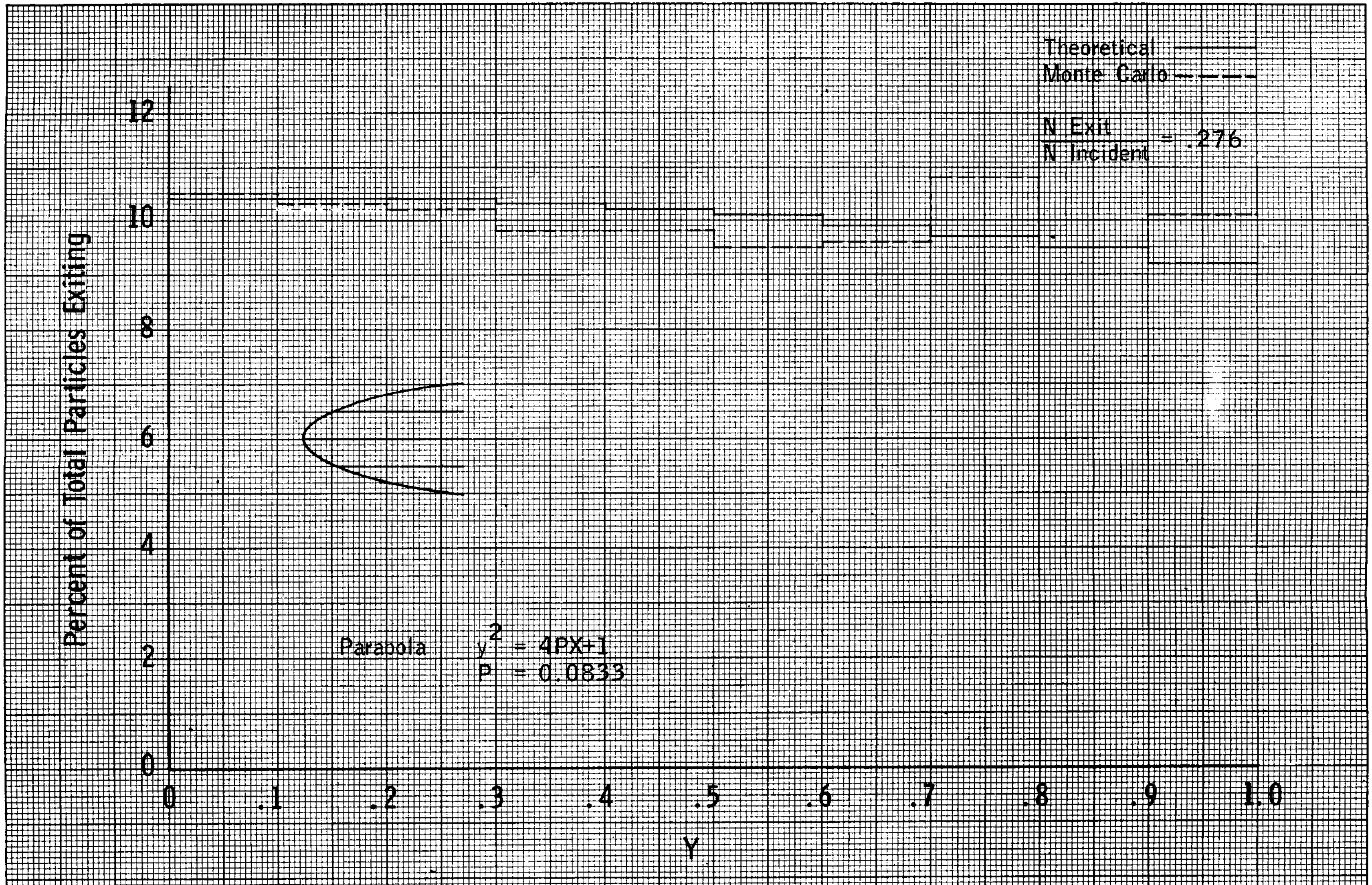

Figure 22 Distribution of Particles Exiting After Single Collision (Incident Distribution Uniform Across Opening)

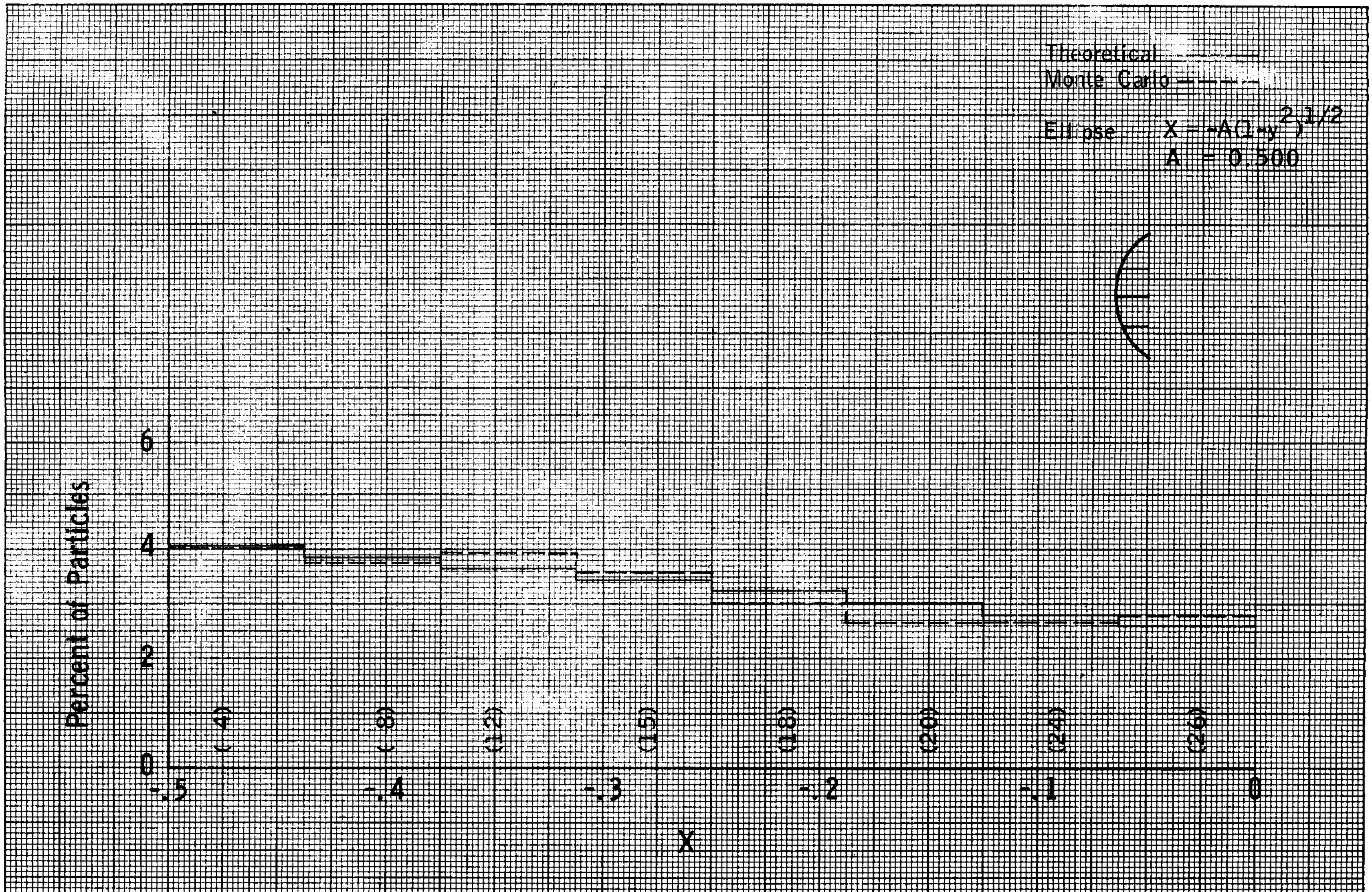

Figure 23 Distribution of Particles Intercepting Center Line After Single Collision
(Incident Distribution Uniform Across Opening)

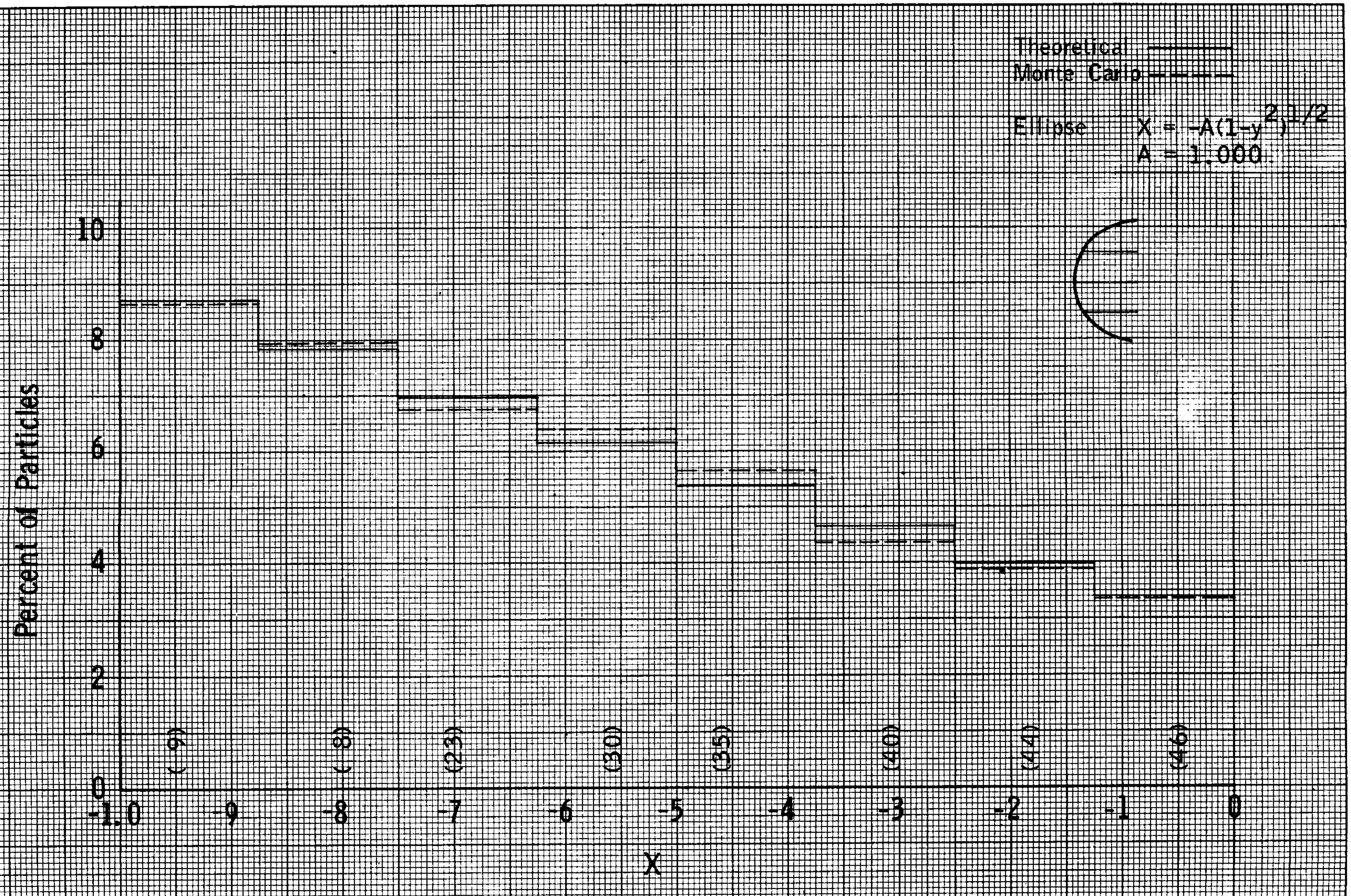

Figure 24 Distribution of Particles Intercepting Center Line After Single Collision
(Incident Distribution Uniform Across Opening)

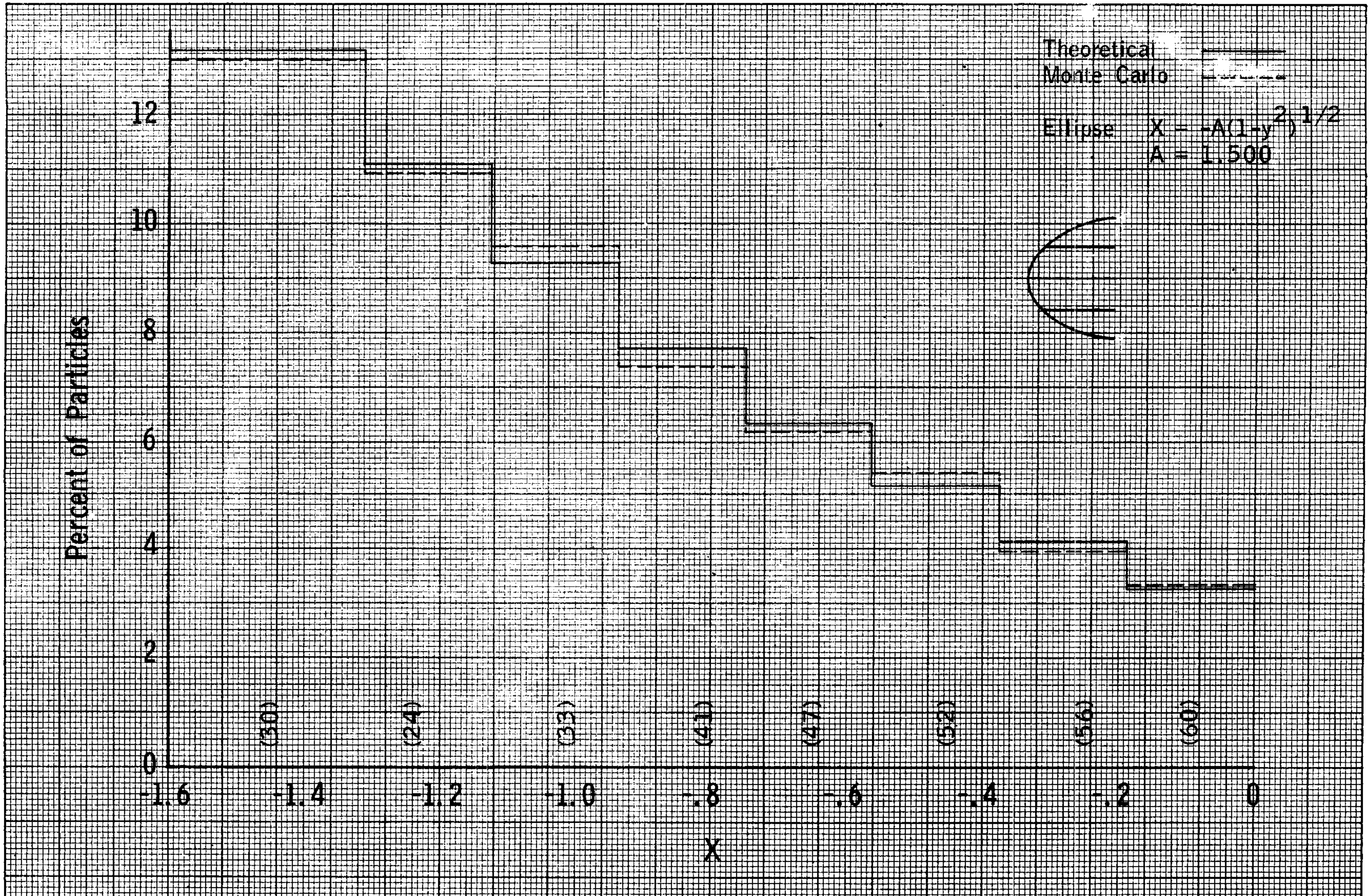

Figure 25 Distribution of Particles Intercepting Center Line After Single Collision (Incident Distribution Uniform Across Opening)

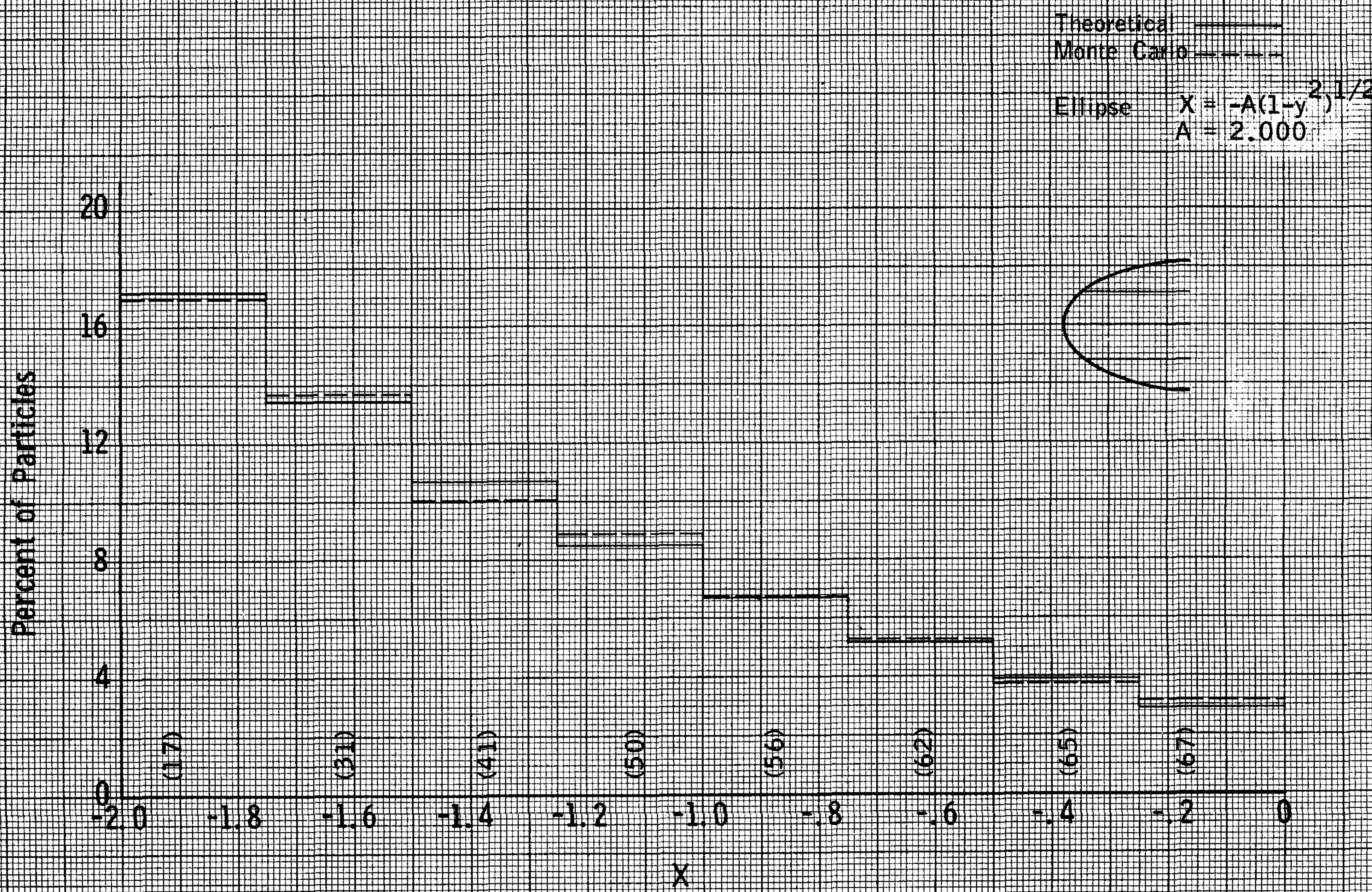

Figure 26 Distribution of Particles Intercepting Center Line After Single Collision (Incident Distribution Uniform Across Opening)

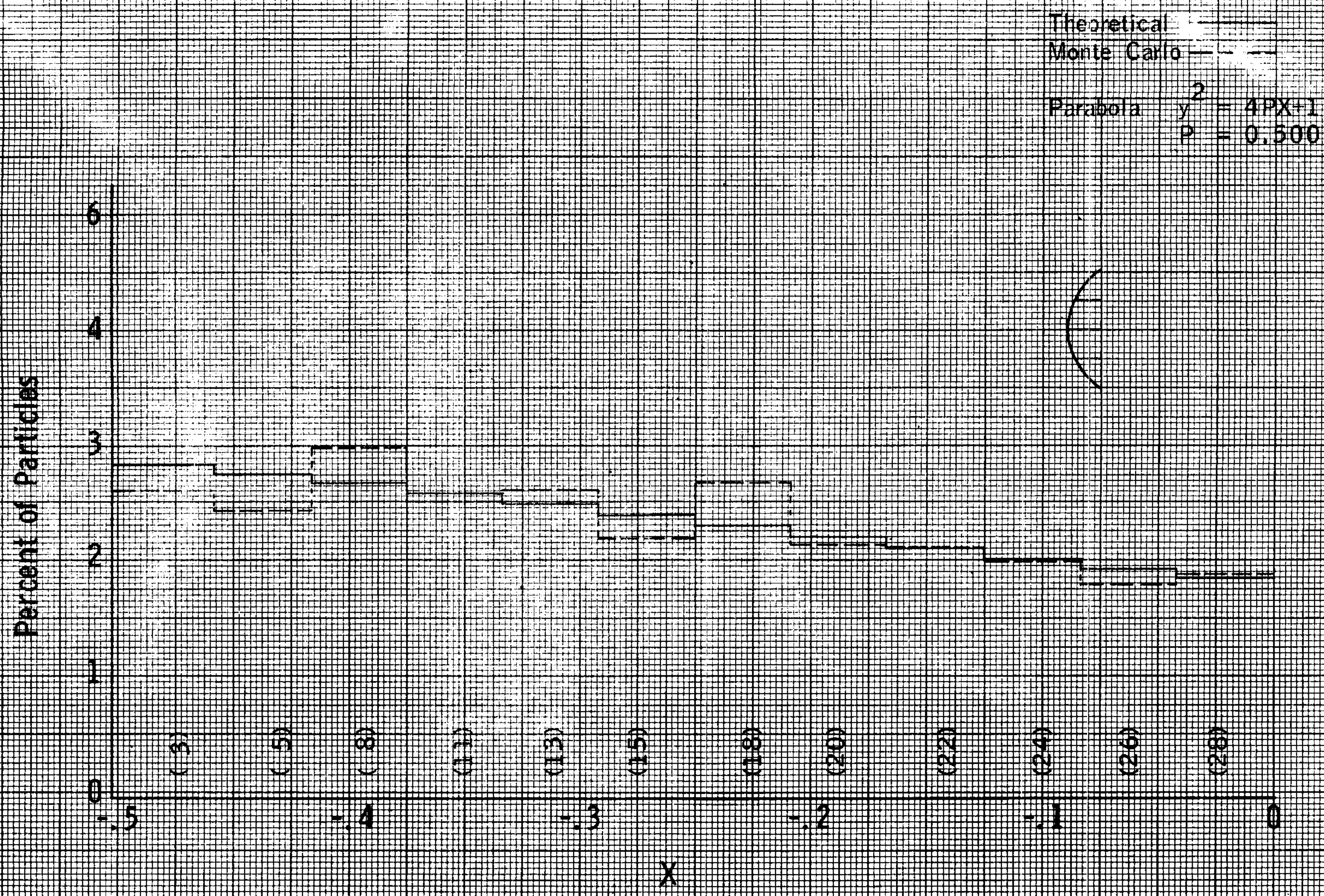

Figure 27 Distribution of Particles Intercepting Center Line After Single Collision
(Incident Distribution Uniform Across Opening)

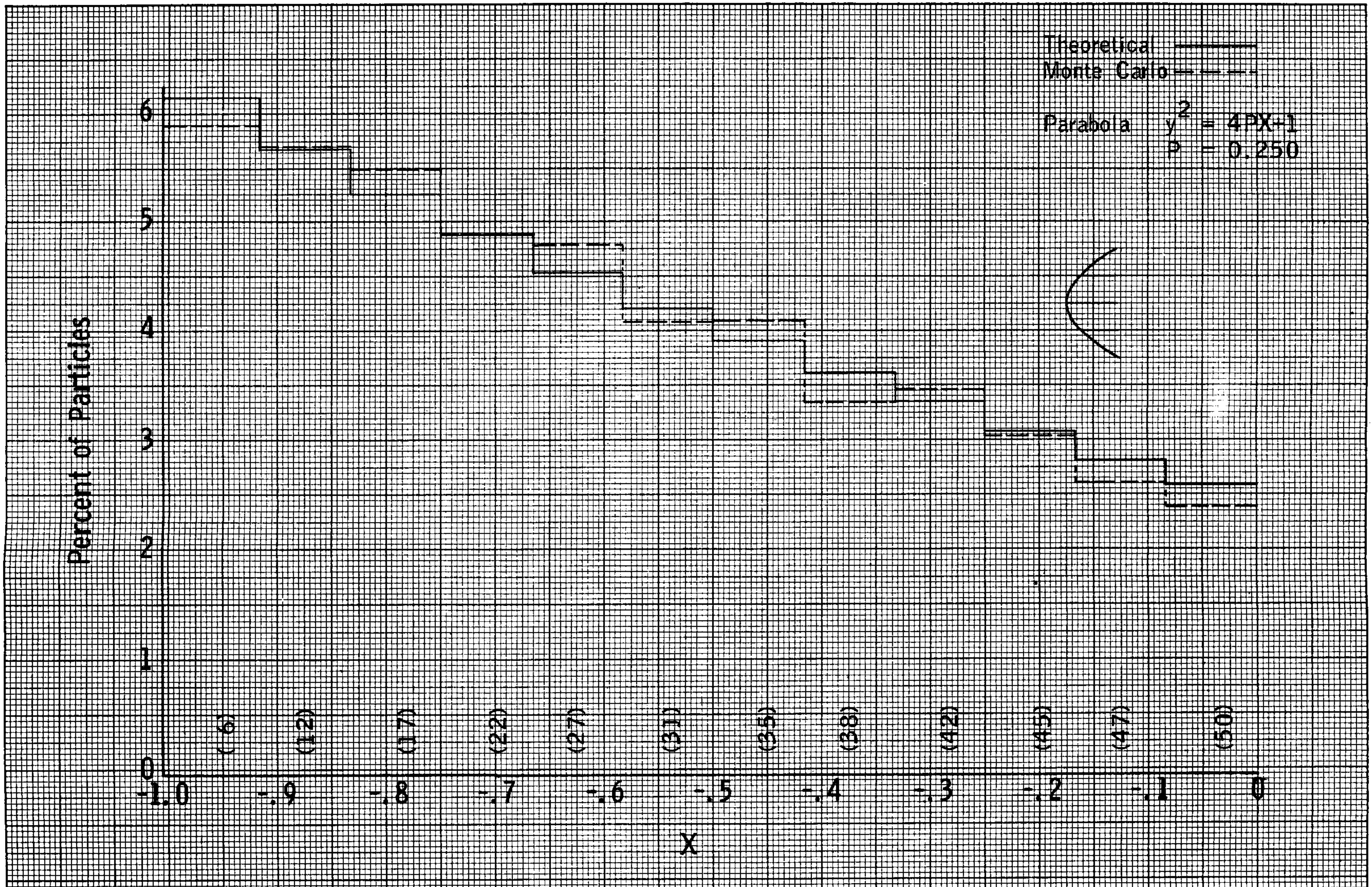

Figure 28 Distribution of Particles Intercepting Center Line After Single Collision (Incident Distribution Uniform Across Opening)

Figure 29 Distribution of Particles Intercepting Center Line After Single Collision (Incident Distribution Uniform Across Opening)

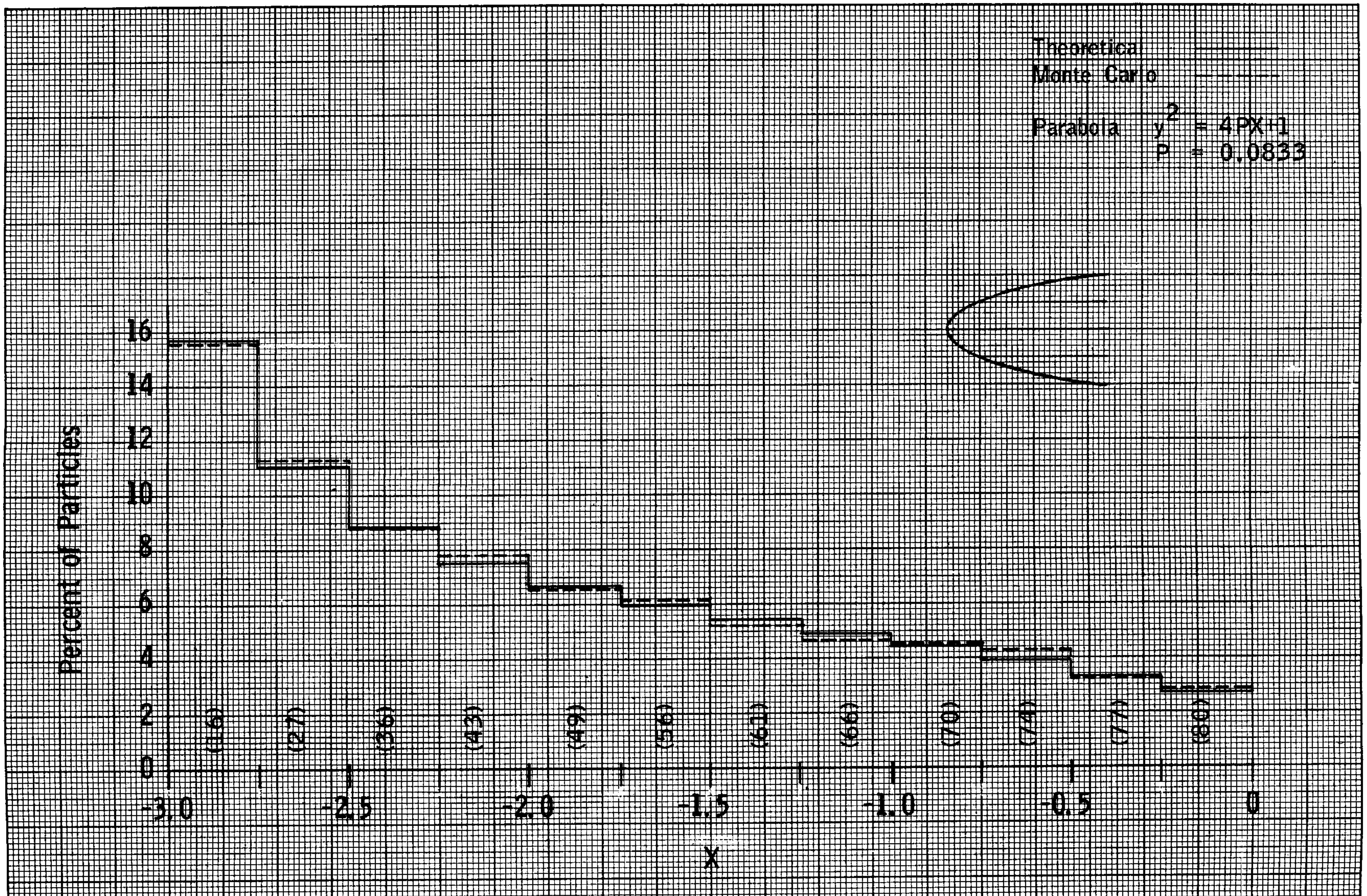

Figure 30 Distribution of Particles Intercepting Center Line After Single Collision
(Incident Distribution Uniform Across Opening)

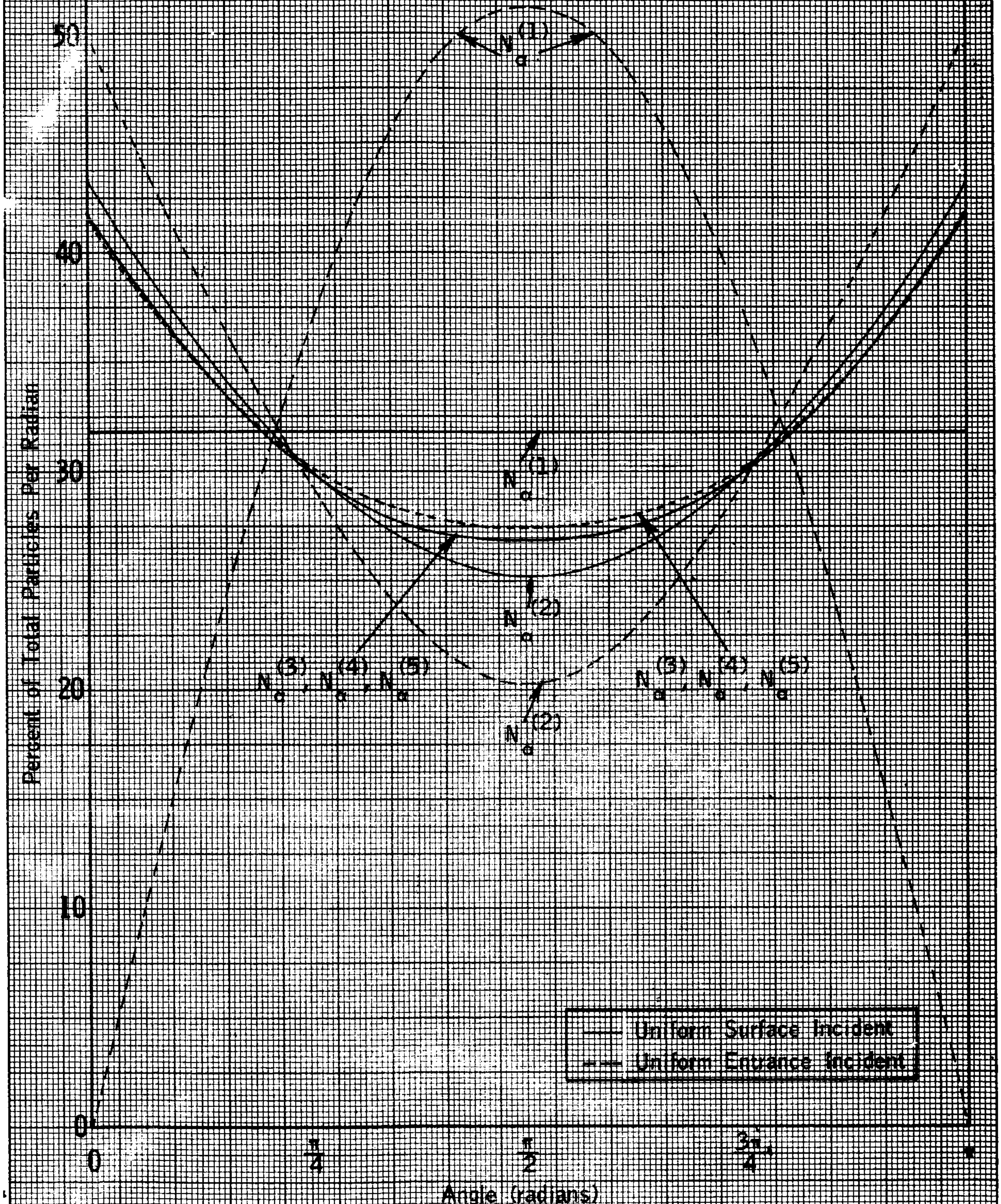

Fig. 31. **Distribution of Particles Over a Semi-Circular Surface at First Five Collisions**

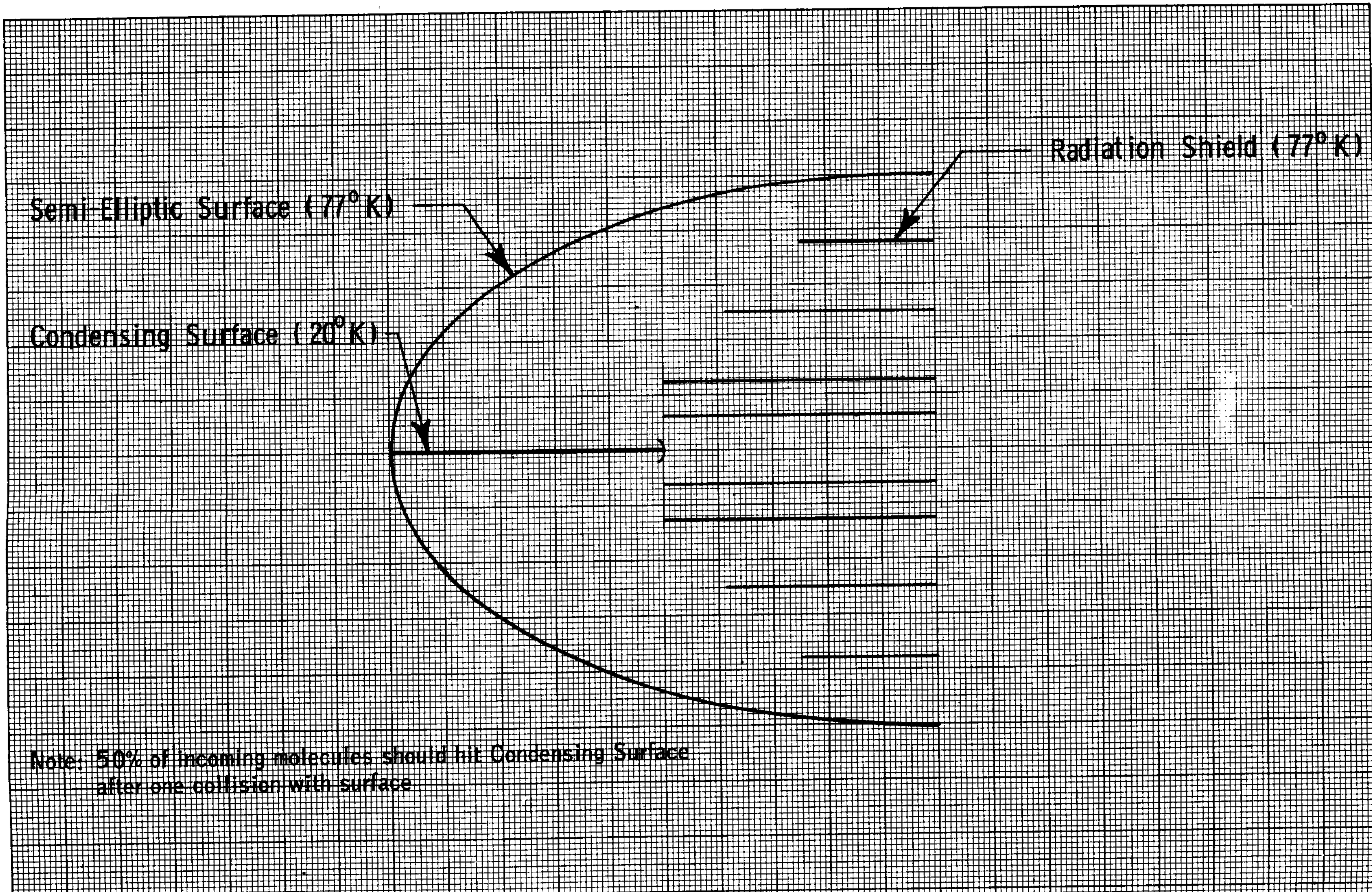

Fig. 32 Sketch of Possible Cryoarray for Low Density Wind Tunnel

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A STUDY OF THE DISTRIBUTION OF MOLECULES UNDER FREE MOLECULAR FLOW CONDITIONS AFTER COLLISIONS WITH SIMPLE GEOMETRIES

DAVID W. TARBELL and JAMES O. BALLANCE

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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